Lect 16. Special Relativity (II)

§1 the relativistic velocity addition

$$\begin{pmatrix} \Delta x' \\ c\Delta t' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} \Delta x \\ c\Delta t \end{pmatrix} \Rightarrow v_x' = \frac{dx'}{dt'} = \frac{\gamma dx - \gamma\beta \cancel{dt}^{\,c}}{-\gamma\beta \frac{dx}{c} + \gamma dt} = \frac{dx/dt - \beta}{-\beta \frac{dx}{dt} + 1}$$

$$= \frac{v_x - v}{1 - \frac{v_x v}{c^2}}$$

$$v_y' = \frac{dy'}{dt'} = \frac{dy}{-\gamma\beta\frac{dx}{c} + \gamma dt} = \frac{v_y}{\gamma\left(1 - \frac{v_x v}{c^2}\right)}, \quad v_z' = \frac{v_z}{\gamma\left(1 - \frac{v_x v}{c^2}\right)}.$$

if $v_x = c$, $v_y = v_z = 0$ $\Rightarrow v_x' = \dfrac{c - v}{1 - \frac{cv}{c^2}} = c$, light velocity doesn't change!

if $v_x, v < c$ $\quad \dfrac{v_x'}{c} = \dfrac{\frac{v_x}{c} - \frac{v}{c}}{1 - \frac{v_x v}{c^2}} = \dfrac{\beta_x - \beta}{1 - \beta_x\beta}$,

$$\left.\begin{array}{l} (1-\beta_x)(1+\beta) > 0 \Rightarrow 1 - \beta_x\beta > \beta_x - \beta \\ (1+\beta_x)(1-\beta) > 0 \Rightarrow 1 - \beta_x\beta > \beta - \beta_x \end{array}\right\} \Rightarrow 1 - \beta_x\beta > |\beta_x - \beta|$$

$$\Rightarrow \left|\frac{v_x'}{c}\right| < 1.$$

§2. 4-vector

$x^\mu = (x_1, x_2, x_3, ct)$, $\rightarrow$ inner product $x^\mu x_\mu = x_1^2 + x_2^2 + x_3^2 - (ct)^2$

$x_\mu = (x_1\ x_2\ x_3, -ct)$ $\quad x_\mu = g_{\mu\nu} x^\nu$, $x^\mu = g^{\mu\nu} x_\nu$ $\quad g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$

4-vector q^μ, transforms as $q^\mu = \Lambda^\mu{}_\nu\, q^\nu$.

$(q_1, q_2\ q_3, q_4)$

$\Lambda^\mu{}_\nu$ is the general Lorentz transformation,

satisfying $\boxed{\Lambda^T g \Lambda = g. \quad \text{or } \Lambda^{\mu'}{}_{\mu}\, g_{\mu'\nu'}\, \Lambda^{\nu'}{}_{\nu} = g_{\mu\nu}.}$

6- degress of freedom.

$$\Lambda: \left[\begin{array}{ccc|c} & R & & \begin{matrix}0\\0\\0\end{matrix} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

spatial part: 3d rotation

3-3degrees of freedom

$$\begin{bmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{bmatrix}$$

space-time boost along x, y, z-directions

3- degrees freedom.

the inner product of 4-vectors is <u>invariant under Lorentz tranform</u>

$$q_1'^\mu\, g_{\mu\nu}\, q_2'^\nu = q_1^{\mu'} \underbrace{\Lambda^\mu{}_{\mu'}\, g_{\mu\nu}\, \Lambda^\nu{}_{\nu'}}_{g_{\mu'\nu'}} q^{\nu'} = q_1^{\mu'}\, g_{\mu'\nu'}\, q_1^{\nu'}.$$

4-velocity

$$u^\mu = \frac{dx^\mu}{d\tau} = \left[\frac{1}{\sqrt{1-\beta^2}}\frac{dx}{dt},\ \frac{1}{\sqrt{1-\beta^2}}\frac{dy}{dt},\ \frac{1}{\sqrt{1-\beta^2}}\frac{dz}{dt},\ \frac{c}{\sqrt{1-\beta^2}}\right]$$

§ Doppler Effect :

quotient rule: if $(\vec{x}, x_4)$ is a 4-vector, and $(\vec{k}, k_4)$ satisfies

$\vec{k}\cdot\vec{x} - k_4 x_4$ is invariant in any frame (Scalar), then $(\vec{k}, k_4)$ is a 4-vector.

Suppose two frames S and S'

$$\underbrace{k^{\mu'}}\, g_{\mu'\nu'}\, x^{\nu'} = k'^{\mu}\, g_{\mu\nu}\, x'^{\nu} = \underbrace{k'^{\mu}\, g_{\mu\nu}\, \Lambda^{\nu}{}_{\nu'}}\, x^{\nu'}$$

$$\Rightarrow k^{\mu'}\, g_{\mu'\nu'} = k'^{\mu}\, g_{\mu\nu}\, \Lambda^{\nu}{}_{\nu'} \qquad g_{\mu'\nu'} = \Lambda^{\mu}{}_{\mu'}\, g_{\mu\nu}\, \Lambda^{\nu}{}_{\nu'}$$

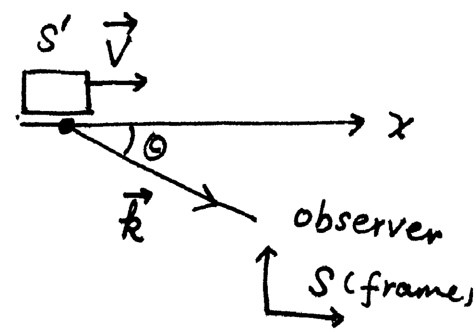
$$k^{\mu'}\, \Lambda^{\mu}{}_{\mu'}\, \underbrace{g_{\mu\nu}\Lambda^{\nu}{}_{\nu'}} = k'^{\mu}\, \underbrace{g_{\mu\nu}\, \Lambda^{\nu}{}_{\nu'}} \Rightarrow \boxed{k'^{\mu} = \Lambda^{\mu}{}_{\mu'}\, k^{\mu'}}$$

$(\vec{k}, k_4)$ is also a 4-vector.

$\phi = A\cos(\vec{k}\cdot\vec{x} - \omega t)$, the phase factor $\vec{k}\cdot\vec{x} - \omega t = k^{\mu} g_{\mu\nu} x^{\nu}$

$\Rightarrow k^{\mu} = (\vec{k}, \frac{\omega}{c})$ is a 4-vector.

In the co-moving frame S', $\Rightarrow k_4' = \frac{\omega_0}{c}$.



$$k_4' = \gamma(k_4 - \beta k_1) = \frac{\omega_0}{c}$$

$$k_1 = |k|\cos\theta = \frac{\omega}{c}\cos\theta$$

$$k_4 = \frac{\omega}{c}$$

$$\Rightarrow \gamma \frac{\omega}{c}[1 - \beta\cos\theta] = \frac{\omega_0}{c}$$

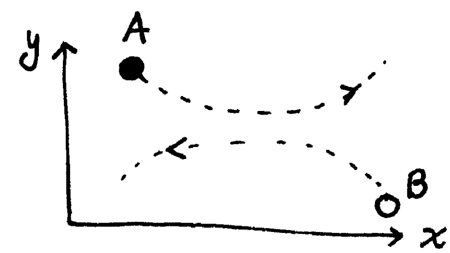
$$\boxed{\omega = \frac{1}{\gamma}\frac{\omega_0}{1 - \beta\cos\theta}}$$

at $\theta = 0, \pi$
$\omega = \frac{1}{\gamma}\frac{\omega_0}{1 \mp \beta}$

at $\theta = \pi/2$
$\omega = \frac{\omega_0}{\gamma}$

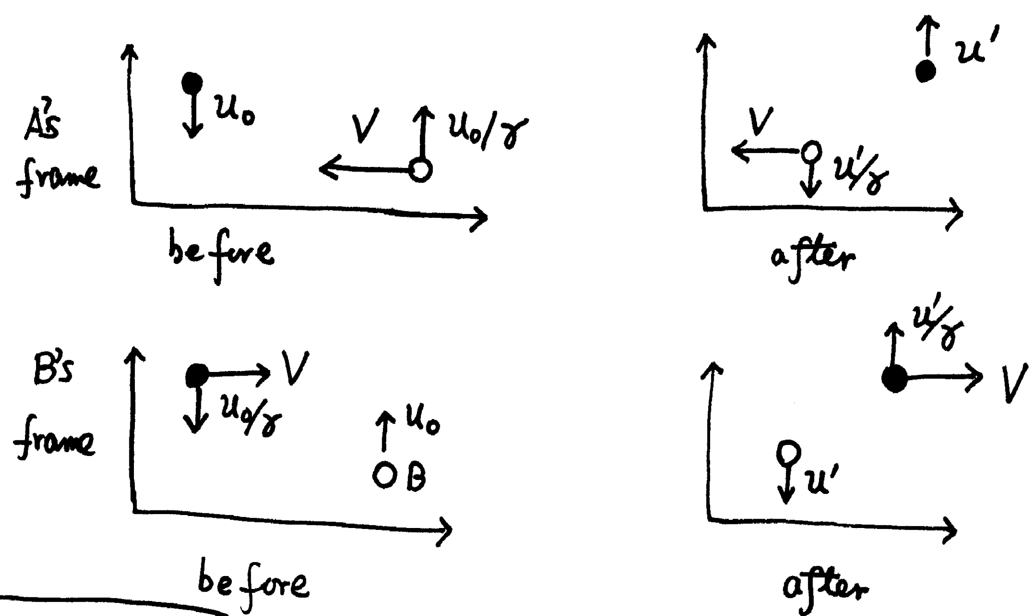
§ relativistic momentum

We want to maintain momentum conservation law. But the classic definition $\vec{P} = m_0 \vec{v}$ does not work any more. We define "m_0" the rest mass measured in the frame where the particle is static.



Consider the following collision process

This collision process in the frames moving with the same horizontal velocity as A, B, respectively.



$\boxed{\text{Before collision:}}$

in B's frame, B's horizontal velocity is zero, vertical velocity u_0.

then in A's frame which has a relative V respect to B's.

then $\quad V_{B,x} = -V, \quad V_{B,y} = \sqrt{1-\left(\frac{V}{c}\right)^2}\, u_0 = \frac{u_0}{\gamma}$

$V_{A,x} = 0, \quad V_{A,y} = -u_0$

After collision:

in A's frame: $V'_{A,x} = 0$, $\quad V'_{A,y} = u'$

$$V'_{B,x} = -V, \quad V'_{B,y} = \frac{u'}{\gamma}, \quad \text{where } \gamma = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}.$$

We seek a conserved quantity similar to classic momentum. We define $\vec{P} = m(v)\vec{V}$, where $m(v)$ is a function of v, and has the unit of mass.

In A's frame, the P_x comes from B. Before collision

$$v_B = \left(V^2 + \frac{u_0^2}{\gamma^2}\right)^{1/2}, \quad \text{and after collision } v_B' = \left(V^2 + \frac{u'^2}{\gamma^2}\right)^{1/2}$$

$$P_x^{\text{before}} = P_x^{\text{end}} \iff m(v_B)V = m(v_B')V$$

$$\Rightarrow \quad u_0 = u'.$$

Now let's write the momentum conservation along y-axis.

$$-m(u_0)u_0 + m\left(\sqrt{V^2+\frac{u_0^2}{\gamma^2}}\right)\frac{u_0}{\gamma} = m(u_0)u_0 - m\left(\sqrt{V^2+\frac{u_0^2}{\gamma^2}}\right)\frac{u_0}{\gamma}$$

$$\iff m\left(\sqrt{V^2+\frac{u_0^2}{\gamma^2}}\right) = \gamma\, m(u_0)$$

set $u_0 \to 0$, we have $m(V) = \gamma\, m(0) = \gamma m_0$ ← rest mass.

thus we define $\vec{P} = \dfrac{m_0 \vec{v}}{\sqrt{1-v^2/c^2}}$ which is conserved.

check $m\left(\sqrt{V^2+\frac{u_0^2}{\gamma^2}}\right) = \gamma\, m(u_0)$

$$\frac{1}{\sqrt{1-\left(V^2+\frac{u_0^2}{\gamma^2}\right)/c^2}} = \frac{\gamma}{\sqrt{\gamma^2-(\gamma^2V^2+u_0^2)/c^2}} = \frac{\gamma}{\sqrt{\gamma^2(1-V^2/c^2)-u_0^2/c^2}}$$

$$= \frac{\gamma}{\sqrt{1-u_0^2/c^2}}$$ ✓ correct!

§ Relativistic energy

we generalize newton's second law $\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}\left[\frac{m_0\vec{v}}{\sqrt{1-\left(\frac{v}{c}\right)^2}}\right]$

define kinetic energy $E_K(v)$

$$E_K(v_b) - E_K(v_a) = \int_a^b \vec{F}\cdot d\vec{r} = \int_a^b \frac{d\vec{p}}{dt}\cdot d\vec{r}$$

$$= \int_a^b \frac{d}{dt}\left(m_0\frac{\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}\right)\cdot\vec{v} = \left.\frac{m_0 v^2}{\sqrt{1-\frac{v^2}{c^2}}}\right|_a^b - \int_a^b \frac{m_0\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}\,d\vec{v}$$

$$\vec{v}\cdot d\vec{v} = \tfrac{1}{2}d(\vec{v}\cdot\vec{v}) = \tfrac{1}{2}dv^2 = v\,dv$$

$$= \left.\frac{m_0 v^2}{\sqrt{1-\frac{v^2}{c^2}}}\right|_a^b - \int_a^b \frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}}\,dv = \left.\frac{m_0 v^2}{\sqrt{1-\frac{v^2}{c^2}}}\right|_a^b + \left. m_0c^2\sqrt{1-\frac{v^2}{c^2}}\right|_a^b$$

set $v_a = 0$

$$\Rightarrow E_K = \frac{m_0 v_b^2}{\sqrt{1-\frac{v_b^2}{c^2}}} + m_0c^2\sqrt{1-\frac{v_b^2}{c^2}} - mc^2 = \underbrace{\frac{m_0c^2}{\sqrt{1-\frac{v_b^2}{c^2}}}}_{\text{total energy}} - \underbrace{m_0c^2}_{\text{rest energy}}$$

$\boxed{E = mc^2}$ more general than just mechanical energy!

§ 4-momentum

We can combine momentum and energy as a 4-vecter.

$$p^{\mu} = \left(\vec{P}, \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}}\right) = \left(\frac{m_0 d\vec{x}/dt}{\sqrt{1-v^2/c^2}}, \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}}\right) = m_0 \left(\frac{d\vec{x}}{d\tau}, \frac{c\,dt}{d\tau}\right)$$

$$d\tau = \sqrt{1-\frac{v^2}{c^2}}\, dt$$

$d\tau$ is the propertime interval for a moving body. 4-momentum is just rest-mass times 4-velocity.

★ For a relastivic collision, all the 4-component must be conserved. Let us look at the 4th component, and consider the non-relativistic limit.

$$E_a^{in} + E_b^{in} = E_a^{fin} + E_b^{fin} \qquad \text{expand to second order of } v^2$$

$$m_{a_0}^{in} c^2 + \frac{1}{2} m_{a,0}^{in} v_a^2 + (a \to b) = m_{a,0}^{fin} c^2 + \frac{1}{2} m_{a,0}^{fin} v_a^{fin^2} + (a \to b)$$

$$M_0^{in} c^2 + T^{in} = M_0^{fin} c^2 + T^{out}$$

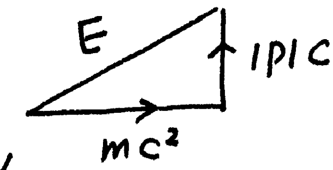
for elastic collision, $T^{in} = T^{out} \Rightarrow$ rest mass are conserved

in elestic collision. $M^{in} \neq M^{fin}$. particles can gain mass by gaining internel energy.

★ useful relations $(\vec{P}, E/c)$

$$\beta = \frac{v}{c} = \frac{\vec{P}c}{E}, \qquad p^2 - \frac{E^2}{c^2} = -(mc)^2 \leftarrow \text{check } (0,0,0,mc)$$

$$E^2 = (pc)^2 + (mc^2)^2$$



$m_e c^2 = 0.5$ Mev, if $T = 0.8$ Mev

$\Rightarrow E = 1.3$ Mev and $pc = 1.2$ Mev $\Rightarrow \beta = \frac{pc}{E} = \frac{1.2}{1.3} \approx 0.92$.

★ Example

① Frank-Hertz experiment: in-elastic collision between electron and Hg atom. Electron looses kinetic energy 4.9 ev (it's rest mass does not change since it's a point particle). This energy is transfered into Hg atom to an excited state. The rest mass of Hg increase $\Delta M = \frac{4.9 \text{ ev}}{c^2} = 8.7 \times 10^{-36}$ kg

$$\frac{\Delta M}{M_{Hg}} \approx 2.6 \times 10^{-11}.$$

② neutron-induced fission

$$n + {}^{235}U \longrightarrow {}^{90}Kr + {}^{143}Ba + 3n$$

$$\Delta T_{kinetic} = 200 \text{ Mev}, \quad \frac{\Delta m}{M} \approx 0.1\%$$

③ particle-antiparticle annihilation

$$e^+ + e^- \rightarrow \gamma$$

(This process can not occur in ~~vaccum~~ free space because of momentum conservation, but can occur for an e^+ collides with an atom).