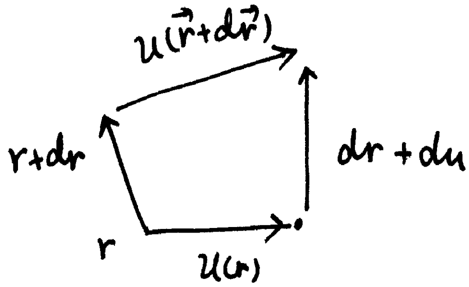
Lect 13. Elasticity of Solids (II)

§ Strain tensor:

Consider a small volume originally at poisition $\vec{r}$, but its new posititin is shifted to $\vec{r} + \vec{u}(r)$. A uniform $\vec{u}(r) = \vec{u}_0$ is just an overall translation, but not distortion. We define the derivative matrix

$$du_i = \sum_j \underbrace{\frac{\partial u_i}{\partial r_j}}_{D_{ij}} dr_j \quad , \quad \text{where } \overleftrightarrow{D} = \begin{bmatrix} \frac{\partial u_x}{\partial r_x} & \cdots & \frac{\partial u_x}{\partial r_z} \\ \vdots & & \vdots \\ \frac{\partial u_z}{\partial r_x} & \cdots & \frac{\partial u_z}{\partial r_z} \end{bmatrix}.$$



However, a rotation will cause non-vanishing D_{ij}, but it's not distortion either!

For a rotation

$$du(r) = \vec{v}\, dt = \vec{\omega}\, dt \times d\vec{r} = d\vec{\theta} \times \vec{r}$$

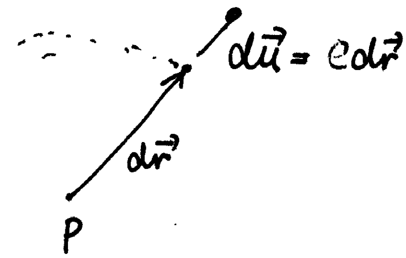
$$D = \begin{bmatrix} 0 & \theta_3 & -\theta_2 \\ -\theta_3 & 0 & \theta_1 \\ \theta_2 & -\theta_1 & 0 \end{bmatrix}$$ which is anti-symmetric.

We decompose D_{ij} into the anti-symmetric part A_{ij} and the symmetric part E_{ij}, where $E_{ij} = \frac{1}{2}(D_{ij} + D_{ji})$.

Example: dilatation

$E_{ij} = e\, I$ → spherical strain → trace-part of E_{ij}

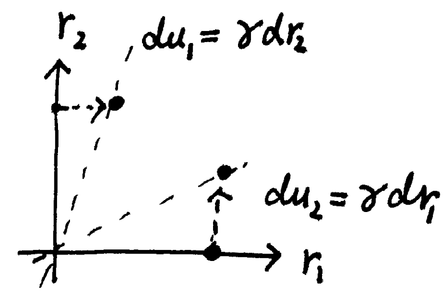
dilatation



expansion & contraction $\quad \frac{dV}{V} = 3e \quad (e \ll 1)$

Example: Shearing strain — trace-less part (off-d

$$E = \begin{bmatrix} 0 & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (\gamma \ll 1), \quad \frac{\partial u_1}{\partial y} = \frac{\partial u_2}{\partial x} = \gamma .$$

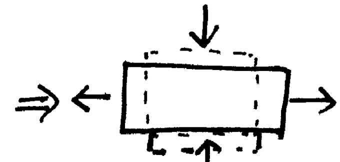


$$\begin{bmatrix} 0 & 0 & \gamma \\ 0 & 0 & 0 \\ \gamma & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \gamma \\ 0 & \gamma & 0 \end{bmatrix}$$

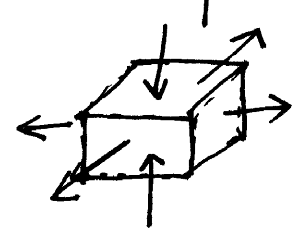
Example: $\begin{bmatrix} \varepsilon_{11} & & \\ & \varepsilon_{22} & \\ & & \varepsilon_{33} \end{bmatrix}$ stretching elements:

dilatation $\quad e = \frac{1}{3} \operatorname{tr} E : \begin{bmatrix} e & & \\ & e & \\ & & e \end{bmatrix}$

$x^2 - y^2$: $\quad \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & -\varepsilon & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$



$x^2 + y^2 - 2z^2 \quad \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & -2\varepsilon \end{bmatrix}$



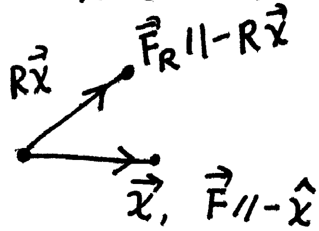
The generalized - Hooke's law

☆ we want to find the relation between the stress tensor Σ, and the strain tensor. The relation should be linear for small strain, and should be rotationally invariant.

$$\boxed{\Sigma(E_R) = \Sigma_R(E)}$$

, i.e., the stress of a rotated strain, should equal to the result of rotating Σ due to the original E.

a simpler example $\qquad \vec{F} = -k\vec{x} \Rightarrow \vec{F}(R\vec{x}) = R\vec{F}(\vec{x})$



i.e. the properity at $R\vec{x}$ can be obtained by the one at $\vec{x}$ through suitable operation.

result of isotropy of space.

The strain tensor can be decomposed into the trace (spherical) & the traceless part

$$\vec{E} = e\vec{I} + \vec{E}'$$

e: $\frac{1}{3}(E_{xx}+E_{yy}+E_{zz})$

$\vec{E}'$: 5-different components

$\frac{1}{2}(E_{xy}+E_{yx})$ $\quad \frac{1}{2}(E_{xx}-E_{yy})$

$\frac{1}{2}(E_{xz}+E_{zx})$ $\quad \frac{1}{\sqrt{3}}(E_{xx}+E_{yy}-2E_{zz})$

$\frac{1}{2}(E_{yz}+E_{zy})$

analogy to 5-d-orbitals

$\vec{I}$ and $\vec{E}'$ transform differently under rotation.

$$\Rightarrow \quad \Sigma = \alpha e I + \beta E' = (\alpha-\beta) e I + \beta E .$$

$$\operatorname{tr}\Sigma = 3\alpha e \Rightarrow e = \frac{\operatorname{tr}\Sigma}{3\alpha}$$

$$\Rightarrow \quad E = \frac{1}{\beta}\left[\Sigma - (\alpha-\beta) e I \right] = \frac{1}{\beta}\Sigma - \frac{\alpha-\beta}{3\alpha\beta}(\operatorname{tr}\Sigma) I .$$

✱ Bulk modulus

$$p = -BM \cdot \frac{dV}{V}$$

$$\Sigma = -pI = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix}$$

$$E = \frac{1}{\beta}(-pI) - \frac{\alpha-\beta}{3\alpha\beta}(-3p) I = -\frac{p}{\alpha} I$$

$$\Rightarrow e = -\frac{p}{\alpha}$$

$$\frac{dV}{V} = 3e = -\frac{3p}{\alpha} \quad \Rightarrow \quad \boxed{BM = \frac{-p}{dV/V} = \frac{-p}{-3p/\alpha} = \frac{\alpha}{3}}$$

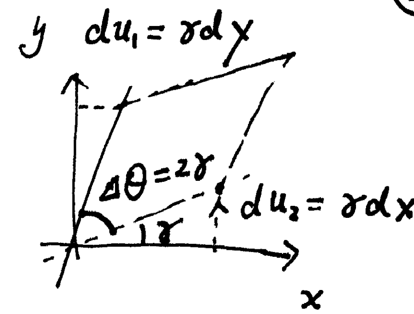
✱ Shear modulus

$$\frac{F}{A} = SM \frac{dy}{dx} = SM \cdot \theta$$

consider the case of $e=0 \Rightarrow \Sigma = \beta E$, $\quad \Sigma = \begin{pmatrix} 0 & \sigma_{12} & 0 \\ \sigma_{12} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \beta \begin{pmatrix} 0 & \epsilon_{12} & 0 \\ \epsilon_{12} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

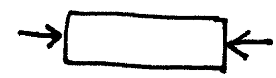
$$\frac{F}{A} = \sigma_{12} = \beta\epsilon_{12} = \beta\gamma$$

$$\frac{F}{A} = \beta\gamma = \beta\frac{\theta}{2} = SM.\theta \Rightarrow \beta = 2\,SM$$



★ Young's modulus

$$YM = \frac{dF/A}{dl/l}$$



$$\Sigma = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow E = \frac{1}{3\alpha\beta}\left[3\alpha\,\Sigma + (\beta-\alpha)\,tr\,\Sigma\,I\right] = \frac{\sigma_{11}}{3\alpha\beta}\begin{bmatrix} 2\alpha+\beta & 0 & 0 \\ 0 & \beta-\alpha & 0 \\ 0 & 0 & \beta-\alpha \end{bmatrix}$$

$$dF/A = \sigma_{11}, \qquad dl/l = \epsilon_{11} = \frac{\sigma_{11}}{3\alpha\beta}(2\alpha+\beta)$$

$$\Rightarrow YM = \frac{\sigma_{11}}{\sigma_{11}(2\alpha+\beta)/(3\alpha\beta)} = \frac{3\alpha\beta}{2\alpha+\beta} = \frac{9\,BM.SM}{3BM+SM}$$