Lect 5: Momentum and angular momentum

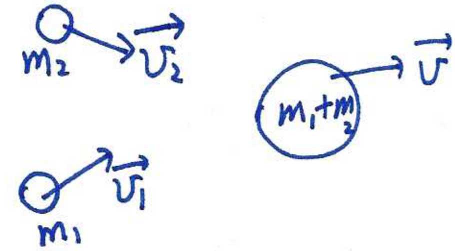
§: Conservation of momentum —— space translation symmetry

- Consider a N-particle system, the total momentum $\vec{P}_t = \sum_{\alpha=1}^{N} \vec{P}_\alpha$ is conserved if the net external force $\vec{F}^{ext}$ is zero. More generally,

$$\boxed{\dot{\vec{P}}_t = \vec{F}^{ext}}$$

Internal force only changes the dirstribution of $\vec{P}_t$ among different particles, but do not change $\vec{P}_t$.

Example: inelastic collision of 2-bodies



$$m_1\vec{v}_1 + m_2\vec{v}_2 = (m_1+m_2)\vec{v}$$

$$\Rightarrow \quad \vec{v} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2}$$

Continuum

$$\boxed{\vec{R} = \frac{1}{M}\int \vec{r}\,dm = \frac{1}{M}\int \rho \vec{r}\,dv}$$

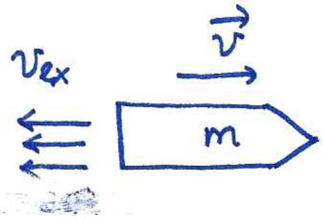
- Center of mass:

$$\vec{R} = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha r_\alpha = \frac{m_1\vec{r}_1 + \cdots m_N\vec{r}_N}{M} \Rightarrow \vec{P}_t = M\dot{\vec{R}}$$

$$\dot{\vec{P}}_t = \vec{F}^{ext} \quad \Rightarrow \quad \boxed{M\ddot{\vec{R}} = \vec{F}^{ext}}$$

$\Rightarrow$ The center of mass motion of a N-particle system is the same as a point particle. This is the reason we can idealize the center of mass motion (translation) of an extended object as mass point motion. For example, the earth motion around the sun.

example: Recoil — rockets



the spent fuel is ejected at speed v_{ex} relative to the rocket.

At time t: $\quad P(t) = mv$

At time $t+\Delta t$: the rocket mass $m+dm$, $\quad (dm<0)$

the spent fuel: $-dm$

$$\Rightarrow P(t+\Delta t) = (m+dm)(\vec{v}+d\vec{v}) - dm\,(v - v_{ex})$$

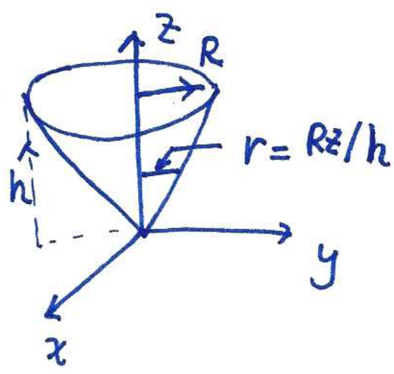
$$\simeq mv + m\,dv + v_{ex}\,dm$$

$$P(t+\Delta t) = P(t) \quad \Rightarrow \quad m\,dv + v_{ex}\,dm = 0$$

$$dv = -v_{ex}\frac{dm}{m} \quad \Rightarrow \quad \boxed{v - v_0 = -v_{ex}\int_{m_0}^{m}\frac{dm'}{m'} = v_{ex}\ln\frac{m_0}{m}}$$

Conside a rocket filled 90% fuel, then $\frac{m_0}{m} = 10 \Rightarrow v - v_0 = v_{ex}\ln 10$
$= 2.3\,v_{ex}$.

2: The CM of a solid cone



$$\vec{R} = \frac{1}{M}\int \rho\,\vec{r}\,dv = \frac{\rho}{M}\int \vec{r}\,dv$$

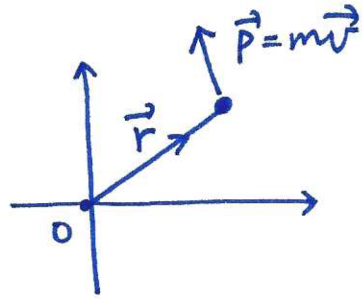
$$R_x = \frac{\rho}{M}\iiint dx\,dy\,dz \cdot x = 0$$

$$R_y = \rho/M \iiint dx\,dy\,dz \; y = 0$$

$$R_z = \frac{\rho}{M}\iiint dx\,dy\,dz\; z = \frac{\rho\int_0^h z\,dz\int_0^{Rz/h} r\,dr\int_0^{2\pi} d\theta}{\rho\int_0^h dz\int_0^{Rz/h} r\,dr\int_0^{2\pi} d\theta} = \frac{\int_0^h dz\,\frac{z}{2}\left(\frac{Rz}{h}\right)^2}{\int_0^h \frac{dz}{2}\left(\frac{Rz}{h}\right)^2} =$$

$$= \frac{\int_0^h z^3\,dz}{\int_0^h z^2\,dz} = \frac{\frac{1}{4}h^4}{\frac{1}{3}h^3} = \boxed{\tfrac{3}{4}h = R_z}$$

§ Angular momentum for a single particle



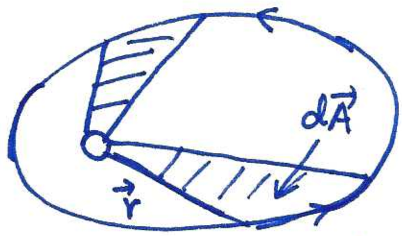
$$\vec{L} = \vec{r} \times \vec{p}$$

$$\dot{\vec{L}} = \dot{\vec{r}} \times \vec{p} + \vec{r} \times \dot{\vec{p}} = \vec{r} \times \vec{F} = \vec{\Gamma}$$

For the central force field, $\vec{F} \parallel \vec{r}$, $\Rightarrow \vec{\Gamma} = 0$,

$\Rightarrow$ angular momentum is conserved.

Example: Kepler's problem



1st law: The planet orbit is planer and closed forming an ellipse.

$d\vec{r} = \vec{v}\,dt$

2nd law: a line draw from the planet to the sun sweeps out equal areas in equal times.

① $\vec{L}$ is conserved: $\longrightarrow$ the orbit is planar. The orbital plane is perpendicular to $\vec{L}$.

② The magnitude of $\vec{L}$: $\vec{L} = \vec{r} \times m\vec{v}$

$$d\vec{A} = \vec{r} \times d\vec{r} \Rightarrow \frac{d\vec{A}}{dt} = \vec{r} \times \vec{v} = \frac{\vec{L}}{m}$$

Kepler's 2nd law is equivlent to angular momentum conservation.

The closed orbit and ellipse requires stronger condition $\rightarrow$ $1/r$ potential or $-1/r^2\,\hat{e}_r$ force.

§ Angular momentum for several particles

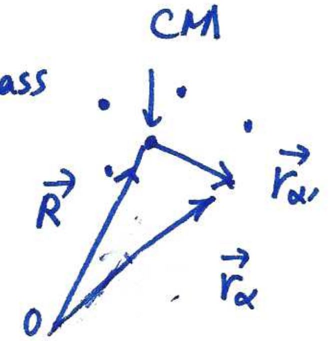
$\alpha = 1, 2 \cdots N$

$\vec{l}_\alpha = \vec{r}_\alpha \times \vec{p}_\alpha$

$\vec{L}_{tot} = \sum_{\alpha=1}^{N} \vec{l}_\alpha = \sum_{\alpha=1}^{N} \vec{r}_\alpha \times \vec{p}_\alpha$

$\vec{R}$: center of mass coordinate

$\vec{r}_{\alpha'}$: center of mass vector



$$\vec{L}_{tot} = \sum_{\alpha=1}^{N} (\vec{R} + \vec{r}_{\alpha'}) \times \vec{p}_\alpha = \vec{R} \times \sum_{\alpha=1}^{N} \vec{p}_\alpha + \sum_{\alpha=1}^{N} \vec{r}_{\alpha'} \times m_\alpha \dot{\vec{r}}_\alpha$$

$$= \vec{R} \times \vec{P}_{tot} + \sum_{\alpha=1}^{N} \vec{r}_{\alpha'} \times m_\alpha (\dot{\vec{r}}_{\alpha'} + \dot{\vec{R}})$$

$$= \vec{R} \times \vec{P}_{tot} + \sum_{\alpha=1}^{N} \vec{r}_{\alpha'} \times m_\alpha \dot{\vec{r}}_{\alpha'} + \underbrace{\left(\sum_{\alpha=1}^{N} m_\alpha \vec{r}_{\alpha'}\right)}_{=0} \times \dot{\vec{R}}$$

please prove it

orbit $\vec{L}_{orbit}$

internal angular momentum with respect to CM : $\vec{L}_{spin}$

$$\boxed{\vec{L}_{tot} = \vec{L}_{orbit} + \vec{L}_{spin}}$$

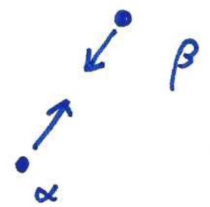
$$\dot{\vec{L}}_{orbit} = \underbrace{\dot{\vec{R}} \times \vec{P}_{tot}}_{=0} + \vec{R} \times \dot{\vec{P}}_{tot} = \vec{R} \times \vec{F}^{ext}$$

$$\dot{\vec{L}}_{tot} = \sum_{\alpha=1}^{N} \vec{r}_\alpha \times (\vec{F}_\alpha^{ex} + \sum_{\beta \neq \alpha} \vec{F}_{\alpha\beta}) = \sum_{\alpha=1}^{N} \vec{r}_\alpha \times \vec{F}_\alpha^{ex} + \sum_{\alpha \neq \beta} \vec{r}_\alpha \times \vec{F}_{\alpha\beta}$$

$$= \sum_{\alpha=1}^{N} \vec{r}_\alpha \times \vec{F}_\alpha^{ex} + \frac{1}{2} \sum_{\alpha \neq \beta} \left(\vec{r}_\alpha \times \vec{F}_{\alpha\beta} + \vec{r}_\beta \times \vec{F}_{\beta\alpha}\right)$$

Due to Newton's 3rd law $\vec{F}_{\alpha\beta} = -\vec{F}_{\beta\alpha}$

$$\sum_{\alpha\neq\beta}\left(\vec{r}_\alpha \times \vec{F}_{\alpha\beta} + \vec{r}_\beta \times \vec{F}_{\beta\alpha}\right) = \underset{\substack{\| \\ 0}}{\sum_{\alpha\neq\beta}(\vec{r}_\alpha - \vec{r}_\beta)\times\vec{F}_{\alpha\beta}}$$



for central force $\vec{F}_{\alpha\beta} \parallel \vec{r}_\alpha - \vec{r}_\beta$ $\Rightarrow$

$$\Rightarrow \dot{\vec{L}}_{tot} = \sum_{\alpha=1}^{N} \vec{r}_\alpha \times \vec{F}_\alpha^{ex} = \vec{\Gamma}^{ex}$$

$$= \sum_{\alpha=1}^{N} (\vec{r}_\alpha' + \vec{R}) \times \vec{F}_\alpha^{ex} = \dot{\vec{L}}_{orbit} + \sum_{\alpha=1}^{N} \vec{r}_{\alpha'} \times \vec{F}_\alpha^{ex}$$

$$\Rightarrow \boxed{\dot{\vec{L}}_{spin} = \sum_{\alpha=1}^{N} \vec{r}_{\alpha'} \times \vec{F}_\alpha^{ex}} \leftarrow \text{turque respect to CM.}$$

In QM, electron has both orbit angular momentum + spin.

spin is quantized at $\hbar/2$.

§ Moment of Inertia (fixed axis rotation)

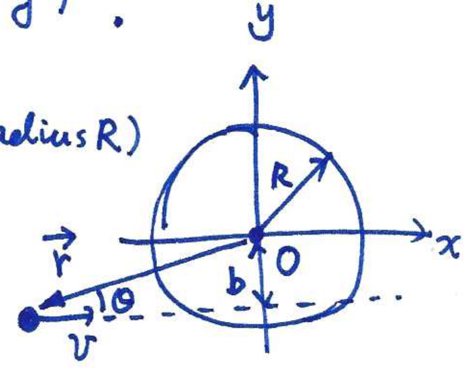
$L_z = I_{zz}\,\omega$, where $\vec{\omega}$ is along z-axis.

$$I_{zz} = \sum_\alpha m_\alpha (x_\alpha^2 + y_\alpha^2) = \int dx\,dy\,dz\;\rho(x^2+y^2) .$$

Examples: ① A uniform circular wheel (mass M, radius R)

A lump of putty (mass m), speed v

with a line of approach of distance b.



$$L_z^{in} = m r v \sin\theta = m r b$$

$$L_z^{fin} = I_{zz}\,\omega = (I_{wheel} + mR^2)\,\omega$$

$$I_{wheel} = \int_0^R \rho\, r^3\, dr\, d\theta = \rho \int_0^{2\pi} d\theta \cdot \frac{R^3}{4} = \frac{\pi}{2}\rho R^3 = \frac{MR^2}{2}$$

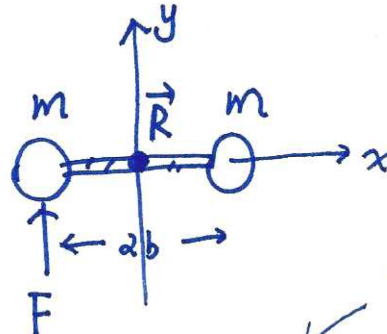
$$M = \int_0^R \rho\, r\, dr\, d\theta = \cdot\ \rho\pi R^2$$

$$\Rightarrow L_z^{fin} = (m + M/2)\,R^2\omega \quad \Rightarrow \omega = \frac{m}{m+M/2}\,\frac{vb}{R^2}$$

② A Sliding and spinning dumbell

At time $t=0$, an moment of impulse

F is acted on the left mass for time Δt.



What will happen?

$\dot{\vec{P}} = F$

$\Rightarrow \Delta P = F\Delta$

The center of mass will move at $\vec{V}_{cm} = \dot{\vec{R}} = \dfrac{F\Delta t}{2m}$

$\vec{F}$ also has torque: with respect to $\vec{R}$

$\vec{\Gamma} = F \cdot b$, $\quad I = 2Mb^2 \quad \Rightarrow \omega = \Gamma^{\Delta t} / I = \dfrac{F\Delta t}{2Mb}$

so $\begin{cases} v_{left} = v_{cm} + \omega b = \dfrac{F\Delta t}{M} \\ v_{right} = v_{cm} - \omega b = 0 \end{cases}$