lect 4 Quadratic air resistance / magnetic cyclontron

$$m\dot{\vec{v}} = m\vec{g} - cv^2\hat{v} = m\vec{g} - c\sqrt{v_x^2+v_y^2}\,\vec{v}, \text{ where } c = \gamma D^2$$

$$\Rightarrow \begin{cases} m\dot{v}_x = -c\sqrt{v_x^2+v_y^2}\,v_x \\ m\dot{v}_y = mg - c\sqrt{v_x^2+v_y^2}\,v_y \end{cases}$$

the motions along x and y directions couple

We first solve two special cases

① no gravity, a body only moves along x-direction. (or gravity is balanced)

By setting $v_y = 0$, we have $m\dot{v}_x = -cv_x^2$

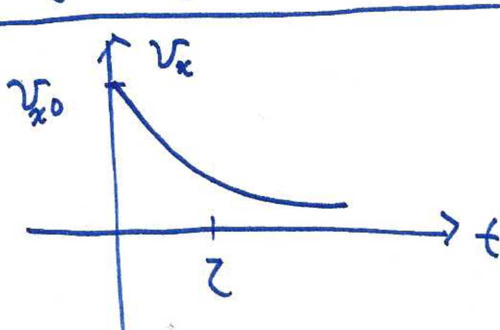
$$\Rightarrow m\frac{dv_x}{v_x^2} = -c\,dt \quad \Rightarrow m\int_{v_x^0}^{v_x(t)} \frac{dv_x}{v_x^2} = -c\int_0^t dt'$$

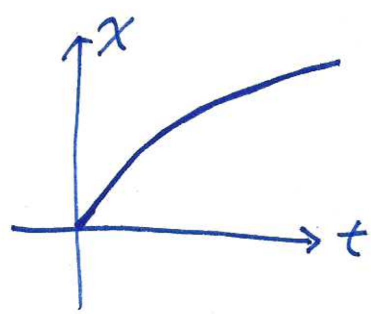
$$m\left(\frac{1}{v_x^0} - \frac{1}{v_x}\right) = -ct \quad \Rightarrow \boxed{v_x(t) = \frac{v_{x,0}}{1 + cv_x^0 t/m} = \frac{v_{x,0}}{1 + t/\tau}}$$

where the time constant $\tau = \frac{m}{c\,v_{x,0}}$

$$\boxed{x(t) = x_0 + \int_0^t v_x(t')\,dt' = v_{x,0}\,\tau \ln(1 + t/\tau)}$$

by setting $x_0 = 0$





Compare with the linear drag, the decay of v_x is slower. It's powerlaw instead of exponetial. Seemingly, x diverges logarithmically. But it cann be realistic. When v is sufficiently small, the drag will become linear.

② vertical motion with $v_x = 0$. fall motion, set down as positive y-direction.

$$\dot{v}_y = g - \frac{c}{m} v_y^2 .$$

Again we solve the terminal velocity

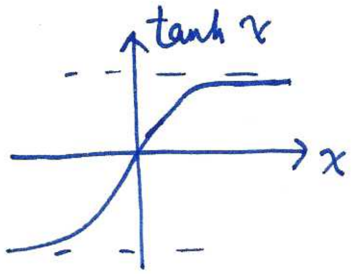
$$v(\infty) = \sqrt{\frac{mg}{c}}$$

then the equation of motion $\Rightarrow$ $\dot{v}_y = g\left(1 - \frac{v_y^2}{v^2(\infty)}\right)$

$$\Rightarrow \frac{dv_y}{1 - \frac{v_y^2}{v^2(\infty)}} = g\,dt \quad \Rightarrow \int_{v_{y,0}}^{v_y} \frac{dv_y}{1 - v_y^2/v^2(\infty)} = gt$$

$\Rightarrow \frac{v(\infty)}{g} \operatorname{arctanh}\left(\frac{v_y}{v(\infty)}\right) = t$. If $v_{y,0} = 0$, we have

$$\operatorname{arctanh}\left(\frac{v_y}{v(\infty)}\right) = \frac{gt}{v(\infty)} \quad \Rightarrow v_y(t) = v(\infty) \tanh \frac{gt}{v(\infty)}$$



$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \simeq \begin{cases} x & x \to 0 \\ 1 & x \to \infty \end{cases}$$

$$\Rightarrow y = \int_0^t dt\, v_y(t) = \frac{v^2(\infty)}{g} \ln\left[\cosh \frac{gt}{v(\infty)}\right]$$

$$\boxed{\cosh x = \frac{e^x + e^{-x}}{2}}$$

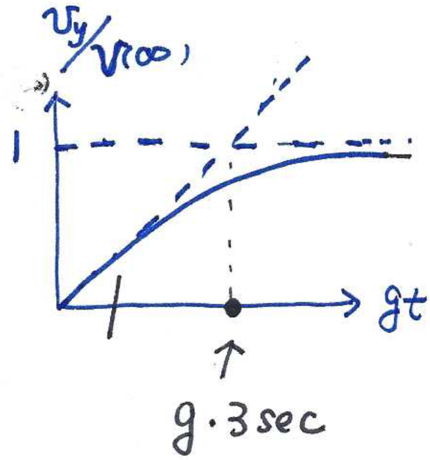
Example: baseball $m = 0.15\,kg$, $D = 7cm$. $\gamma = 0.25\,N\,s^2/m^4$.

When it falls vertically, $v(\infty) = \sqrt{\frac{mg}{\gamma D^2}} = 35\,m/s$.

it's terminal velocity

$$v_y(t) = v(\infty)\tanh\left(\frac{gt}{v(\infty)}\right)$$

$$\approx \begin{cases} gt & \text{at } \frac{gt}{v(\infty)} \ll 1 \\ v(\infty) & \text{at } \frac{gt}{v(\infty)} \gg 1 \end{cases}$$

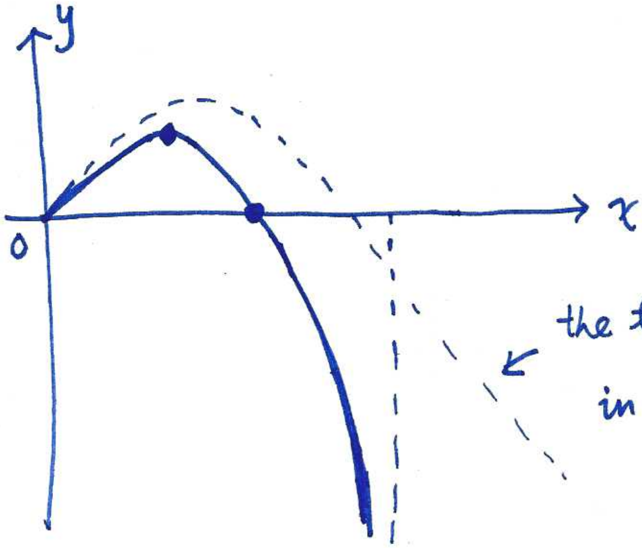


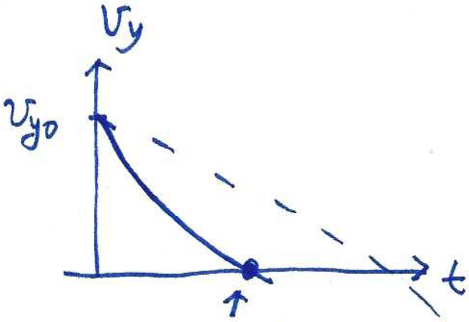
$$y = \frac{v^2(\infty)}{g}\ln\left[\cosh\frac{gt}{v(\infty)}\right] \approx \begin{cases} \frac{1}{2}gt^2 & t \ll \frac{v(\infty)}{g} \\ v(\infty)\,t & \text{at } t \gg \frac{v(\infty)}{g} \end{cases}$$

* Now we study the coupled x and y - motion

$$\begin{cases} \dot{v}_x = -\frac{c}{m}\sqrt{v_x^2+v_y^2}\, v_x \\ \dot{v}_y = -g - \frac{c}{m}\sqrt{v_x^2+v_y^2}\, v_y \end{cases}$$

it cannot be solved analytically. The textbook presents a numeric solution.





Remarks ① The highest height is lowered by the resistance compared to that in the vaccum.

② $v_y = 0$ occurs eariler, i.e. the highest point is reached earier.

③ at $t \to \infty$, the trajectory becomes vertical, and $v_x(\infty) = 0$. Then $v(\infty)$ is the same as in the vertical motion. Then

$$\dot{v}_x \simeq -\frac{c}{m} v(\infty)\, v_x, \quad \text{then } v_x \approx A e^{-t/\tau}, \text{ with } \tau = \frac{m v(\infty)}{c}.$$

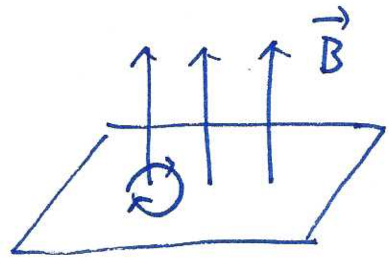
So $v_x \to 0$ exponientially. Thus $x(t) = \int_0^\infty v_x(t)\, dt$

converges, i.e. it can only travel for a finite distance along the x-direction.

§ Motion in a uniform magnetic field

$$\boxed{\vec{F} = \frac{q}{c}\,\vec{v}\times\vec{B}}\,.\qquad \text{set } \vec{B} = B\hat{z}$$

← Gaussian Unit



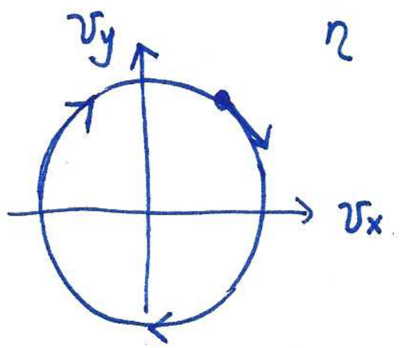
$$\begin{cases} m\dot{v}_x = \frac{q}{c}B\,v_y \\ m\dot{v}_y = -\frac{qB}{c}v_x \\ m\dot{v}_z = 0 \end{cases}$$

$\rightarrow\ \omega = \frac{qB}{mc}$ cyclontron freguency

$$\Rightarrow \begin{cases} \dot{v}_x = \omega v_y \\ \dot{v}_y = -\omega\, v_x \end{cases} \Rightarrow v_x + iv_y = -i\,\omega(\dot{v}_x + i\dot{v}_y)$$

Define $\eta = v_x + iv_y \Rightarrow \boxed{\eta = -i\omega\dot{\eta}}$

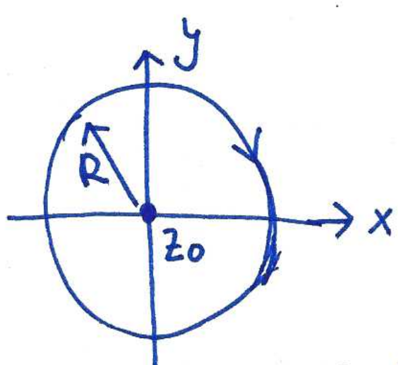
$$\Rightarrow \eta = A e^{-i\omega t} \quad \text{with } A = v_x(0) + iv_y(0)$$



we also define $z = x + iy$

$$\Rightarrow z = \text{const} + \int^t \eta\, dt = z_0 + \frac{iA}{\omega}e^{-i\omega t}$$

↑ center of the circle.



The cyclontron radius $R = \left|\frac{A}{\omega}\right| = \frac{vmc}{qB}$.

$\Rightarrow$ momentum magnetude $P = \frac{qBR}{c}$.

$\rightarrow$ orbial angular momentum $L = PR = \frac{qBR^2}{c}$

Classically, the circular orbital can be at any size.

But quantum mechanically, it has a minimal size.

The orbital angular momentum $L_{min} = \frac{qBR^2}{c} = \hbar$

$$\Rightarrow R = \sqrt{\frac{\hbar c}{qB}}$$ ← the cyclontron radius.