Lect 17 conservation laws, Lagragian for magnetic forces

§ Generalized momenta and ignorable coordinates

Try to relate $\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)$ with Newton's 2nd law $F_i = \frac{d}{dt} P_i$.

The left hand side can be analogous to force, and we define

$$\boxed{P_i = \frac{\partial L}{\partial \dot{q}_i}},$$

and if L doesn't depend on q_i

$$\Rightarrow \boxed{P_i \equiv \text{const}} \leftarrow \dot{P}_i = \frac{\partial L}{\partial q_i} = 0.$$

Coordinates q_i not appearing in L are called ignorable or cyclic. The associated momenta P_i's are conserved.

§ Noether's theorem — symmetries $\Leftrightarrow$ conservation laws

1. Conservation of total momentum

if $L(\vec{r}_1, \cdots \vec{r}_N, \dot{\vec{r}}_1 \cdots \dot{\vec{r}}_N)$ is invariant under a global translation $\vec{r}_\alpha \to \vec{r}_\alpha + \vec{\epsilon}$, then $\delta L = \frac{\partial L}{\partial \vec{r}_1} \cdot \vec{\epsilon} + \cdots + \frac{\partial L}{\partial \vec{r}_N} \cdot \vec{\epsilon} = 0$

$$\Rightarrow \sum_{\alpha=1}^{N} \frac{\partial L}{\partial \vec{r}_\alpha} = 0.$$

Then $\frac{d}{dt}\vec{P}_\alpha = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}_\alpha}\right) = \frac{\partial \mathcal{L}}{\partial \vec{r}_\alpha}$

$$\frac{d}{dt}\left(\sum_{\alpha=1}^{N}\vec{P}_\alpha\right) = \sum_{\alpha=1}^{N}\frac{\partial \mathcal{L}}{\partial \vec{r}_\alpha} = 0$$

2. energy conservation

$$\frac{d}{dt}\mathcal{L}[q_i, \dot{q}_i, t] = \sum_i \frac{\partial \mathcal{L}}{\partial q_i}\dot{q}_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial \mathcal{L}}{\partial t}$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \dot{P}_i$$

$$\Rightarrow \frac{d}{dt}\mathcal{L} = \sum_i\left(\dot{P}_i\dot{q}_i + P_i\ddot{q}_i\right) + \frac{\partial \mathcal{L}}{\partial t} = \frac{d}{dt}\sum_i(P_i\dot{q}_i) + \frac{\partial \mathcal{L}}{\partial t}$$

$$\Rightarrow \frac{d}{dt}\left[\sum_i P_i\dot{q}_i - \mathcal{L}\right] = -\frac{\partial \mathcal{L}}{\partial t}$$

we define Hamiltonian $\boxed{H = \sum_i P_i\dot{q}_i - \mathcal{L}}$, and we have $\boxed{\frac{d}{dt}H = -\frac{\partial \mathcal{L}}{\partial t}.}$. If $\mathcal{L}$ does not explicitly depend on time, then H is conserved!

we can check that if the generalized $\underbrace{\text{coordinates}}_{(q_1 \cdots q_n)}$ are related to the Cartesians in a time-indepedent away, i.e.

$$\vec{r}_\alpha = \vec{r}_\alpha (q_1 \cdots q_n),$$

then $H = T + \mathcal{U}$.

Proof: $T = \sum_\alpha \frac{1}{2} m_\alpha \dot{\vec{r}}_\alpha^{\,2}$

$$\dot{\vec{r}}_\alpha = \sum_{i=1}^{n} \frac{\partial \vec{r}_\alpha}{\partial q_i} \dot{q}_i \Rightarrow \dot{\vec{r}}_\alpha^{\,2} = \sum_{j,k} \left(\frac{\partial \vec{r}_\alpha}{\partial q_j} \dot{q}_j \right) \cdot \left(\frac{\partial \vec{r}_\alpha}{\partial q_k} \dot{q}_k \right)$$

$$= \sum_{jk} \dot{q}_j \dot{q}_k \left(\frac{\partial \vec{r}_\alpha}{\partial q_j} \cdot \frac{\partial \vec{r}_\alpha}{\partial q_k} \right)$$

$$\Rightarrow \boxed{T = \frac{1}{2} \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha^{\,2} = \frac{1}{2} \sum_{jk} A_{jk} \dot{q}_j \dot{q}_k} \text{ with}$$

$$A_{jk} = A_{jk}(q_1 \cdots q_n) = \sum_\alpha m_\alpha \frac{\partial \vec{r}_\alpha}{\partial q_j} \cdot \frac{\partial \vec{r}_\alpha}{\partial q_k}$$

Then $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} = \sum_j A_{ij} \dot{q}_j$

$$\sum_i p_i q_i = \sum_i \left(\sum_j A_{ij} \dot{q}_j \right) \dot{q}_i = 2T$$

$$\Rightarrow \boxed{\mathcal{H} = 2T - (T - \mathcal{U}) = T + \mathcal{U}.}$$

§ Lagrangian for magnetic forces (velocity - dependent forces)

The lorentz force is velocity-dependent, which cannot be expressed as gradient of scalar potential. Then how to put it in the formalism of Lagrangian to yield $\vec{F} = q(\vec{E} + \frac{\vec{v}}{c} \times \vec{B})$?

check $\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)$ and $\mathcal{L} = T - U$.

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = - \frac{\partial U}{\partial q_i} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right).$$

If we use cartisian coordinates, the left hand side is just the force $m\vec{a}$. The $-\frac{\partial U}{\partial q_i}$ is the force due to the usual scalar potential, and we need to design the velocity dependence of U. We want

$$\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right) = - \nabla_r U + \frac{d}{dt}\left(\frac{\partial U}{\partial \vec{v}}\right) = m\vec{a}.$$

From E& M, we learn
$$\begin{cases} \vec{E} = - \nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \\ \vec{B} = \nabla \times \vec{A} \end{cases}$$

$$\Rightarrow \vec{F} = q\left(-\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} + \frac{1}{c}(\vec{v}\times\nabla\times\vec{A})\right)$$

$$(\vec{v}\times\nabla\times\vec{A})_x = v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - v_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)$$

$$= v_y\frac{\partial A_y}{\partial x} + v_z\frac{\partial A_z}{\partial x} + v_x\frac{\partial A_x}{\partial x} - \left(v_y\frac{\partial A_x}{\partial y} + v_z\frac{\partial A_x}{\partial z} + v_x\frac{\partial A_x}{\partial x}\right)$$

$$= \vec{v}\cdot\partial_x\vec{A} - (\vec{v}\cdot\vec{\partial})A_x \quad \leftarrow \quad \frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial}{\partial\vec{r}}\frac{d\vec{r}}{dt} = \frac{\partial}{\partial t} + \vec{v}\cdot\frac{\partial}{\partial\vec{r}}$$

$$\Rightarrow (\vec{v}\times\vec{\nabla}\times\vec{A})_x = \frac{\partial}{\partial x}(\vec{v}\cdot\vec{A}) - \left(\frac{d}{dt} - \frac{\partial}{\partial t}\right)A_x$$

$$\Rightarrow F_x = q\left[-\frac{\partial}{\partial x}\left(\phi - \frac{1}{c}\vec{v}\cdot\vec{A}\right) - \frac{1}{c}\frac{d}{dt}\left(\frac{\partial}{\partial v_x}(\vec{A}\cdot\vec{v})\right)\right]$$

$$\downarrow \frac{1}{c}\frac{d}{dt}A_x$$

since A only depend on $\vec{r}$

$$\Rightarrow \text{define } U = q\phi - \frac{q}{c}\vec{A}\cdot\vec{v}$$

we have

$$\boxed{\vec{F} = -q\nabla U + \frac{d}{dt}\left(\frac{\partial U}{\partial\vec{v}}\right)}$$

and $$\mathcal{L} = T - q\left[\phi - \frac{1}{c}\vec{A}\cdot\vec{v}\right]$$

or $$\boxed{\mathcal{L} = T - \left[q\phi - \frac{1}{c}\vec{j}\cdot\vec{A}\right]}$$ where $\vec{j} = q\vec{v}$