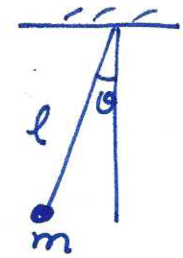
Lect 15: Lagrangian formalism for constrained systems

Examples



① pendulum: If use cartisen coordinate, its x and y are not independent, satisfying $x^2+y^2=\ell^2$.

The actual degree of freedom is $2-1=1$.

we can use the generalized coordinate θ. to solve the constraint.

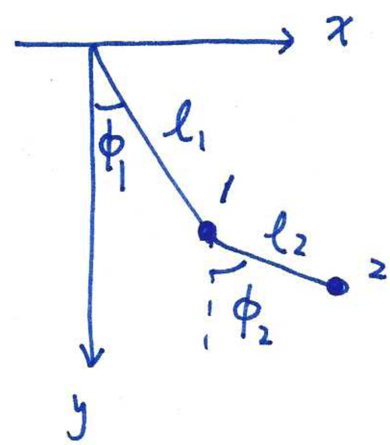
$$\mathcal{L} = T - \mathcal{U} = \frac{1}{2} m \ell^2 \dot\phi^2 - mg\ell(1-\cos\phi)$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\phi}\right) \Rightarrow -mg\ell \sin\phi = \frac{d}{dt}(m\ell^2\dot\phi) = m\ell^2\ddot\phi$$

$$\boxed{\ddot\phi = -\frac{g}{\ell}\sin\phi}$$

② Consider a N-particle system, the $\vec{r}_i$ $(i=1,\cdots N)$ can be expressed in terms of $q_1, \cdots q_n$ — generalized coordinates.

we can treat $q_1, \cdots q_n$ as independent variables.



Example: $x_1 = \ell_1 \sin\phi_1$, $\quad x_2 = \ell_1 \sin\phi_1 + \ell_2 \sin\phi_2$

$y_1 = \ell_1 \cos\phi_1 \quad\quad y_2 = \ell_1\cos\phi_1 + \ell_2 \cos\phi_2$

$$\Rightarrow T = \frac{1}{2} m_1 \ell_1^2 \dot\phi_1^2 + \frac{1}{2} m_2 \left[(\ell_1 \cos\phi_1 \dot\phi_1 + \ell_2 \cos\phi_2 \dot\phi_2)^2 + (-\ell_1 \sin\phi_1 \dot\phi_1 - \ell_2 \sin\phi_2 \dot\phi_2 \right.$$

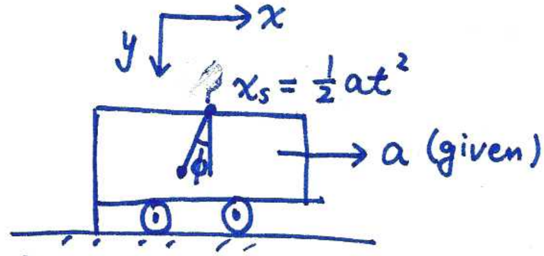
$$= \frac{1}{2} m_1 \ell_1^2 \dot\phi_1^2 + \frac{1}{2} m_2 \left[\ell_1^2 \dot\phi_1^2 + \ell_2^2 \dot\phi_2^2 + 2\ell_1\ell_2 \dot\phi_1 \dot\phi_2 \cos(\phi_1 - \phi_2) \right]$$

$$U = -m_1 g l_1 \cos\phi_1 - m_2 g(l_1\cos\phi_1 + l_2\cos\phi_2)$$

$$\Rightarrow \quad L = \frac{1}{2}(m_1+m_2) l_1^2 \dot\phi_1^2 + \frac{1}{2} m_2 l_2^2 \dot\phi_2^2 + m_2 l_1 l_2 \dot\phi_1\dot\phi_2 \cos(\phi_1-\phi_2)$$

$$+ (m_1+m_2) g l_1 \cos\phi_1 + m_2 g l_2 \cos\phi_2$$

③ time-dependent generalized coordinate



$$\begin{cases} x = \frac{1}{2}at^2 - l\sin\phi \\ y = l\cos\phi \end{cases}$$

$$T = \frac{1}{2}m(\dot x^2 + \dot y^2) = \frac{m}{2}\left[(at - l\cos\phi\,\dot\phi)^2 + l^2\sin^2\phi\,\dot\phi^2\right)$$

$$= \frac{m}{2}\left[a^2t^2 + l^2\dot\phi^2 - 2al\cos\phi\,\dot\phi\, t\right]$$

$$U = -mgl\cos\phi$$

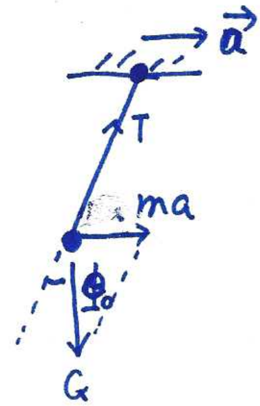
$$\Big\} \quad L = T - U$$

$$\frac{\partial L}{\partial \phi} = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot\phi}\right]$$

$$\cancel{mal\sin\phi\,\dot\phi\, t} - mgl\sin\phi = \frac{d}{dt}\left[ml^2\dot\phi - mal\cos\phi\; t\right]$$

$$= ml^2\ddot\phi + \cancel{mal\sin\phi\,\dot\phi\, t} - mal\cos\phi$$

$$\Rightarrow \quad \boxed{-\frac{g}{l}\sin\phi + \frac{a}{l}\cos\phi = \ddot\phi}$$



New equilibrium position

$$\tan\phi_0 = \frac{ma}{G} = a/g$$

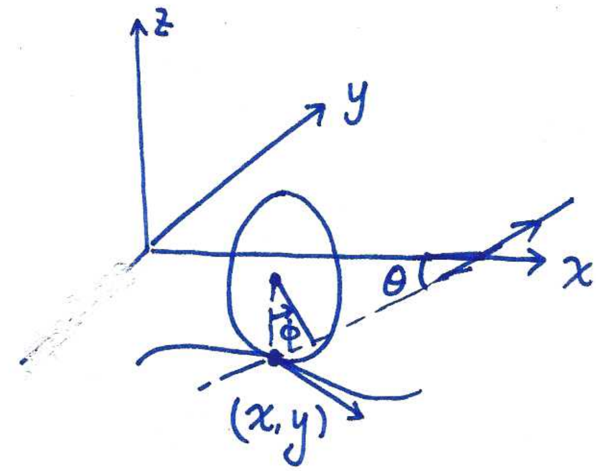
§ Holonomic constraints

In some situations, it's not easy to find independent generalized coordinates to solve the constraint. Rather the constraint can be expressed as equations involving coordinates

$$\boxed{f(\vec{r}_1, \vec{r}_2, \cdots, t) = 0.}$$

These kind of constraints are called holonomic.

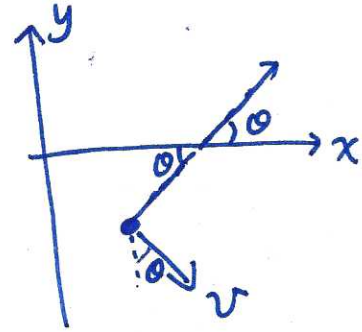
In fact, there're indeed non-holonomic constraints — for common examples. Consider a disk rolling on the xy-plane



- Angle of rotation ϕ
- (x, y) of the center
- θ: between the disk axis and the x-axis

But we know there're only two degrees of freedom

$$\begin{cases} v = R\dot{\phi} \\ \dot{x} = v\sin\theta \\ \dot{y} = -v\cos\theta \end{cases} \Rightarrow \begin{cases} \dfrac{dx}{d\phi} = a\sin\theta \\ \dfrac{dy}{d\phi} = -a\cos\theta \end{cases}$$



Please note the angle appearing on the RHS is θ, and thus this set of differential equations are non-integrable. This is a kind of non holonomic constraint. — The relation between x y and the ration angle is path-dependent!

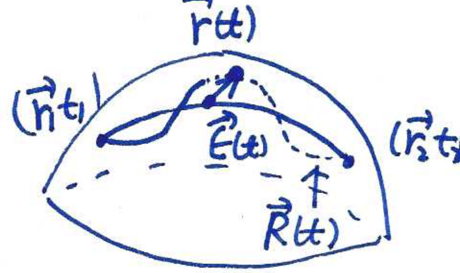
★ For holonomic constraints, the least action principle and the Lagrangian equations still apply. For constraint systems, there're forces due to constraint denoted by $\vec{F}_{cstr}$. These forces are typically unknown — we only known their consequences, i.e. the constraints, often only after we solve the motion, we can obtain them but not before we solve the problem.

Other forces are typically known, such as gravity. They are denoted as $\vec{F}$, and they are expressed as $\vec{F} = -\nabla U(\vec{r}, t)$.

The total force $\vec{F}_{tot} = \vec{F}_{cstr} + \vec{F}$. Nevertheless, the Lagrangien $\mathcal{L} = T - \mathcal{U}$, in which "$\mathcal{U}$" only counts the $\vec{F}$, but not $\vec{F}_{cstr}$.

Eliminating the constraint forces in the Lagrangian formalism is the major advantage. Below we shall see that actually it still obey the least action principle. We will use a single particle problem as an example, where the motion is confined on a surface.

Conside a particle passes $\vec r_1$ at t_1, and $\vec r_2$ at t_2. We denote $\vec r(t)$ as the actual path that particle travels, and $\vec R(t)$ is a wrong path lying in the neighborhood of $\vec r(t)$. And both $\vec R(t)$ and $\vec r(t)$ lie on the surface : $\vec R(t) = \vec r(t) + \vec \epsilon(t)$. As $\vec\epsilon(t)$ becomes infinitsimal, the $\vec\epsilon(t)$ lies on the tangient plane.



The action associated with $\vec R(t)$

$$S = \int_{t_1}^{t_2} \mathcal{L}(\vec R, \dot{\vec R}, t)\, dt$$

$$\delta S = \int_{t_1}^{t_2} \delta \mathcal{L}\, dt \quad \text{with } \delta\mathcal{L} = \mathcal{L}(R, \dot R, t) - \mathcal{L}(r, \dot r, t)$$

$$= \frac{1}{2} m\left[(\dot r + \dot\epsilon)^2 - \dot r^2\right] - \left[U(\vec r + \vec\epsilon) - U(\vec r)\right]$$

$$\approx m\dot{\vec r}\cdot\dot{\vec\epsilon} - \vec\epsilon \cdot \nabla U$$

$$\Rightarrow \delta S = \int_{t_1}^{t_2} \left[m\dot{\vec r}\cdot\dot{\vec\epsilon} - \vec\epsilon\cdot\nabla U\right] dt = -\int_{t_1}^{t_2} \vec\epsilon \cdot \left[m\ddot{\vec r} + \nabla U\right] dt$$

$$m\ddot{\vec r} = \vec F_{cstr} + \vec F = \vec F_{cstr} - \nabla U$$

$$\delta S = -\int_{t_1}^{t_2} \vec{\epsilon} \cdot \vec{F}_{cstr}\, dt$$

Since $\vec{F}_{cstr}$ lies in the normal direction, while $\vec{\epsilon}$ lies in the tangetial plane. $\Rightarrow \vec{\epsilon} \perp \vec{F}_{cstr} \Rightarrow \boxed{\delta S = 0.}$

With the holonomic constraint, (the surface equation), the degrees of freedom is reduced to two. If we can find two generalized coordinates q_1, q_2 to solve the constraint, we can express

$S = \int_{t_1}^{t_2} \mathcal{L}(q_1, q_2, \dot{q}_1, \dot{q}_2)\, dt$, which leads to $\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)$ for $i=1,2$.

§ Lagrange multipliers and constraint forces

In many case, it's not easy to solve the constraint explicitly. We need to work with non-independent variables, which are subject to the constraint equation $f_i(\vec{r}_1, \vec{r}_2 \cdots ; t) = 0$. Let's consider a simplest example. The system has two coordinates x and y, but they are subject to the constraint $f(x, y) = 0$.

The action is still written as $S = \int_{t_1}^{t_2} \mathcal{L}(x, y, \dot{x}, \dot{y})\, dt$.

Consider variations of the path

$$\left.\begin{array}{l} x(t) \to x(t) + \delta x(t) \\ y(t) \to y(t) + \delta y(t) \end{array}\right\} \Rightarrow \delta S = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial \dot{x}} \delta \dot{x} + \frac{\partial L}{\partial y} \delta y + \frac{\partial L}{\partial \dot{y}} \delta \dot{y} \right) dt$$

Again $\delta S = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \right) \delta x(t)\, dt + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} \right) \delta y(t)\, dt$

$= 0$

for any variation $\delta x(t)$ and $\delta y(t)$ satisfying the constraint.

Since $f(x,y) = 0 \Rightarrow \boxed{\delta f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y = 0} \leftarrow$ differential form

we can multiply this constraint by $\lambda(t) \leftarrow$ the Lagrange multiplier

and add to δS

$$\Rightarrow \delta S = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x} + \lambda(t)\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \right) \delta x\, dt$$

$$+ \int_{t_2}^{t_2} \left(\frac{\partial L}{\partial y} + \lambda(t)\frac{\partial f}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} \right) \delta y\, dt = 0.$$

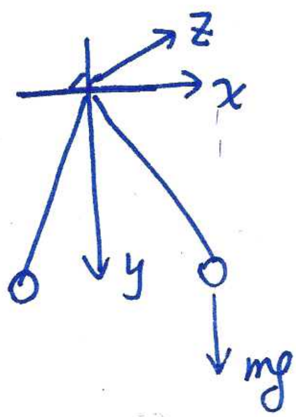
We choose $\lambda(t)$ is such a way that

$$\boxed{\frac{\partial L}{\partial x} + \lambda(t)\frac{\partial f}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}}$$ along the actual path.

Then the same $\lambda(t)$ must also make because $\delta S = 0$.

$$\boxed{\frac{\partial \mathcal{L}}{\partial y} + \lambda(t)\frac{\partial f}{\partial y} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}}}$$

Now we have three varible x, y, and λ to solve, — we also have the constraint Eq. $\boxed{f(x, y) = 0}$.



Example: pedulum

$$\begin{cases} \mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - U(x, y) \\ f(x, y) = x^2 + y^2 - \ell^2 = 0 \end{cases}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial \mathcal{L}}{\partial x} + \lambda \frac{\partial f}{\partial x} \Rightarrow \begin{cases} m\ddot{x} = -\frac{\partial U}{\partial x} + \lambda \frac{\partial f}{\partial x} \\ m\ddot{y} = -\frac{\partial U}{\partial y} + \lambda \frac{\partial f}{\partial y} \end{cases}$$

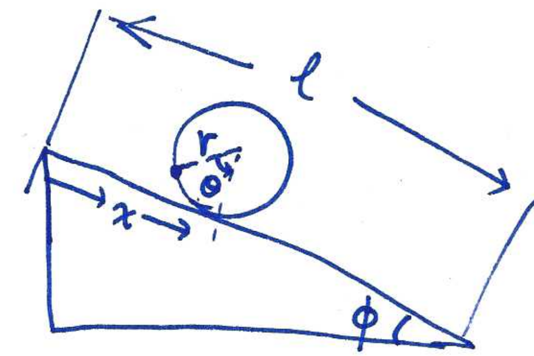
$\Rightarrow$ $\lambda \frac{\partial f}{\partial x}$, $\lambda \frac{\partial f}{\partial y}$ are the x and y componets of the constraint forces.

plug in $U = -mgy$ $\Rightarrow$

$$\begin{cases} m\ddot{x} = 2\lambda x & ① \\ m\ddot{y} = -mg + 2\lambda y & ② \\ x^2 + y^2 = \ell^2 \end{cases}$$

$$\Rightarrow \quad m y \ddot{x} = m x \ddot{y} - m g x$$

$$\Rightarrow \frac{d}{dt}[m(x\dot{y} - \dot{y}x)] = mgx \quad \Rightarrow \boxed{\frac{d}{dt} L_z = \Gamma}$$



Example: a hoop rolling without slipping

The generalized coordinate x, and θ

but they're not independant.

$$r\,d\theta = dx \quad \Rightarrow \quad r\delta\theta = \delta x \rightarrow \lambda[\delta x - r\delta\theta]$$

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}Mr^2\dot{\theta}^2, \quad V = Mg(l-x)\sin\phi$$

$$\Rightarrow L = T - V = \frac{M}{2}\dot{x}^2 + \frac{Mr^2}{2}\dot{\theta}^2 - Mg(l-x)\sin\phi$$

$$\Rightarrow \delta S = \int_{t_1}^{t_2}\left[\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \lambda\right]\delta x + \left[\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \lambda r\right]\delta\theta$$

$$\Rightarrow \begin{cases} \dfrac{\partial L}{\partial x} + \lambda = \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{x}} \\ \dfrac{\partial L}{\partial \theta} - \lambda r = \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{\theta}} \end{cases} \Rightarrow \begin{cases} M\ddot{x} - Mg\sin\phi = \lambda \\ Mr^2\ddot{\theta} = -\lambda r \end{cases}$$

From the constraint $r\ddot{\theta} = \ddot{x} \Rightarrow M\ddot{x} = Mr\ddot{\theta} = -\lambda$

$$\Rightarrow M\ddot{x} + \lambda = 2M\ddot{x} = Mg\sin\phi \Rightarrow \begin{cases} \ddot{x} = \dfrac{g\sin\phi}{2}, \quad \ddot{\theta} = \dfrac{g\sin\phi}{2r}. \\ \lambda = -\dfrac{Mg\sin\phi}{2} \end{cases}$$