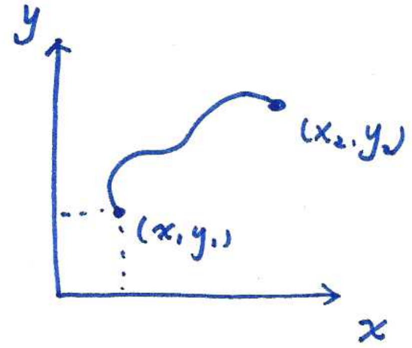
Lect 13 Calculus of Variations

§1. Functional — function of function

Example ① Among all the curves connecting points 1 and 2. Find the path $y(x)$ with the shortest length.

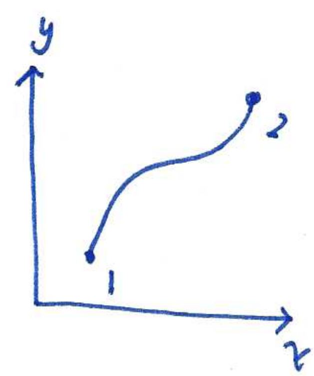


$$ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1+(y'(x))^2}$$

$$\Rightarrow \quad L = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1+(y'(x))^2}\, dx = L[y(x)].$$

② Fermat's principle

In a medium with the distribution of refraction index $n(x)$. The path of light propergation is related to the time it spents. minimal



$$\text{time of travel} = \int_1^2 dt = \int_1^2 \frac{ds}{v}$$

$$v = c/n$$

$$\Rightarrow T[y(x)] = \int_1^2 \frac{n(x,y)}{c}\, ds = \frac{1}{c}\int_1^2 n(x, y(x))\sqrt{1+(y'(x))^2}\, dx$$

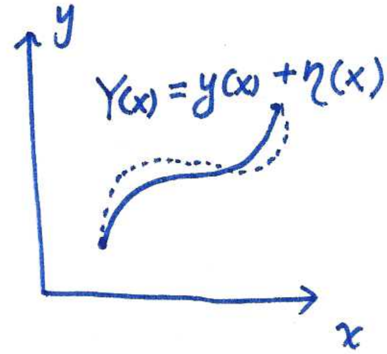
§2. Minimization (extremum) of functional

For function $f(x)$, $\rightarrow$ $\left.\frac{\partial f(x)}{\partial x}\right|_{x=x_0} = 0$, $\rightarrow$ stationary point, saddle point.

• The Euler-Lagrange equation:

$$S = \int_{x_1}^{x_2} f(y(x), y'(x), x)\, dx,$$

$y(x)$ is an as-yet unknown curve jointing (x_1, y_1) and (x_2, y_2). $y(x)$ satisfies $y(x_1)=y_1$ and $y(x_2)=y_2$.



Consider a path $y(x)$, and a small variation

$Y(x) = y(x) + \eta(x)$, where $\eta(x)$ is small,

and $\eta(x_1) = \eta(x_2) = 0$.

Introducing a series of curves, $Y(x;\alpha) = y(x) + \alpha\eta(x)$. Then

$$S(\alpha) = \int_{x_1}^{x_2} f(y+\alpha\eta,\ y'+\alpha\eta',\ x)\, dx$$

$$\frac{\partial f}{\partial \alpha} = \eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'}$$

$$\Rightarrow \frac{dS(\alpha)}{d\alpha} = \int_{x_1}^{x_2}\left[\eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'}\right] dx$$

$$\int_{x_1}^{x_2} \eta' \frac{\partial f}{\partial y'}\, dx = \int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\, d\eta = \left(\frac{\partial f}{\partial y'}\eta\right)\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) dx$$

$\eta(x_1) = \eta(x_2) = 0$

$$\Rightarrow \frac{dS(\alpha)}{d\alpha} = \int_{x_1}^{x_2} \eta(x) \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} \right] = 0.$$

if $y(x)$ is the saddle point solutin, or, extremum solutin, the $dS(\alpha)/d\alpha = 0$. This andition should be satisfied fir arbitary path $\eta(x)$, thus

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0}$$

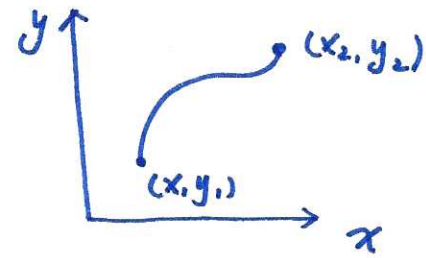
Euler - Lagrange equation

If fir arbitary functin $g(x)$, we have $\int_{x_1}^{x_2} \eta(x) g(x) = 0$, then $g(x) \equiv 0$.

otherwise, we take $g(x) = \eta(x)$, then $\int_{x_1}^{x_2} \eta(x) g(x) = \int_{x_1}^{x_2} |g(x)|^2 dx = 0$

$\Rightarrow g(x) \equiv 0.$

§ Applications



- Shortest length problem

$$L[y(x)] = \int_{x_1}^{x_2} f(y, y', x)\, dx$$

$$f(y, y', x) = (1 + y'^2)^{1/2} \rightarrow \frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial y'} = \tfrac{1}{2}(1+y'^2)^{-1/2}\, 2y'$$

$$\Rightarrow \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0 \Rightarrow \frac{d}{dx}\left[(1+y'^2)^{-1/2} y'\right] = 0 \Rightarrow$$

$$\frac{y'}{(1+y'^2)^{1/2}} = c \Rightarrow (y')^2 = c(1+(y')^2) \Rightarrow (y')^2 = \text{const}$$

or $y' = \text{const}$, or $y(x)$ is a linear function
$\longrightarrow$ straight line.

- we can also use parameter equation $x(u)$ and $y(u)$, then

$$ds = \left[(x'(u))^2 + (y'(u))^2\right]^{1/2} du$$

$$\Rightarrow L = \int_{u_1}^{u_2} (x'^2 + y'^2)^{1/2}\, du.$$ For general problems, we consider

the variation of $S = \int_{u_1}^{u_2} f[x(u), y(u), x'(u), y'(u)]\, du$

By similar method as before, we have

$$\boxed{\frac{\partial f}{\partial x} - \frac{d}{du}\frac{\partial f}{\partial x'} = 0, \quad \frac{\partial f}{\partial y} - \frac{d}{du}\frac{\partial f}{\partial y'} = 0.}$$

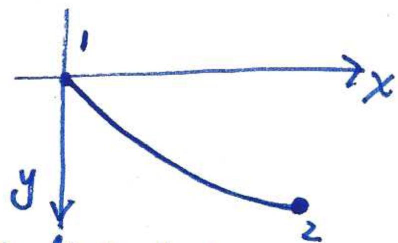
In this case, f is independent on x and y $\Rightarrow$

$$\begin{cases} \dfrac{d}{du}\dfrac{\partial f}{\partial x'} = 0 \\ \dfrac{d}{du}\dfrac{\partial f}{\partial y'} = 0 \end{cases} \Rightarrow \left.\begin{array}{l} \dfrac{x'}{\sqrt{(x')^2+(y')^2}} = C_1 \\ \dfrac{y'}{\sqrt{(x')^2+(y')^2}} = C_2 \end{array}\right\} \Rightarrow \frac{y'}{x'} = \text{const}$$

or $dy/dx = \text{const}$.

§ Brachisto-chrone

$$\text{time}(1 \to 2) = \int_1^2 \frac{ds}{v}$$



The curve with shortest length is a straight line, but if the initial part of the curve has steeper slope, the roller can develop a quick speed eariler. As a result, the actual time spent can be smaller.

$$ds = \sqrt{1+\frac{dx}{dy}}\, dy \quad \Rightarrow \text{time}(1 \to 2) = \frac{1}{\sqrt{2g}} \int_0^{y_2} \frac{\sqrt{x'^2(y)+1}}{\sqrt{y}}\, dy$$

$$f(x, x', y) = \frac{(x'^2+1)^{1/2}}{y^{1/2}}$$

$$\frac{\partial f}{\partial x} = \frac{d}{dy}\frac{\partial f}{\partial x'} = 0 \Rightarrow \frac{\partial f}{\partial x'} = \frac{1}{2}\frac{(x'^2+1)^{-1/2}\, 2x'}{y^{1/2}} = \text{const}$$

$$\frac{x'^2}{y(1+x'^2)} = \text{const} = \frac{1}{2a}$$

"a" carries the unit of length

$$\Rightarrow \quad x' = \sqrt{\frac{y}{2a-y}}$$

Choose the down direction as the positive direction $\Rightarrow \frac{dx}{dy} > 0$

$$\frac{x}{a} = \int \frac{dy}{a} \sqrt{\frac{y/a}{2 - y/a}}$$

plug in variable transformation

$$y/a = 1 - \cos\theta$$

at the initial position $y = 0$

with $\theta_1 = 0$

$$\frac{x}{a} = \int \sin\theta \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \, d\theta$$

$$= \int \sin\theta \tan\frac{\theta}{2} \, d\theta$$

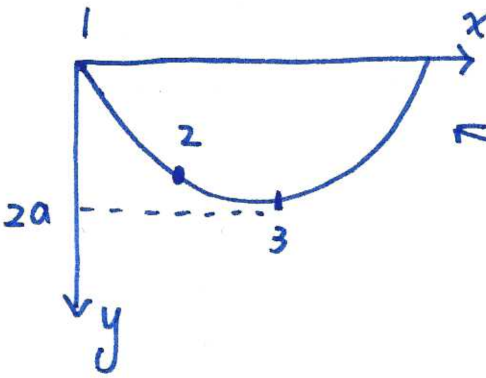
$$= \int 2\sin^2\frac{\theta}{2} \, d\theta = \int (1 - \cos\theta) \, d\theta$$

$$= \theta - \sin\theta$$

$$\Rightarrow \begin{cases} x = a[\theta - \sin\theta] \\ y = a[1 - \cos\theta] \end{cases}$$

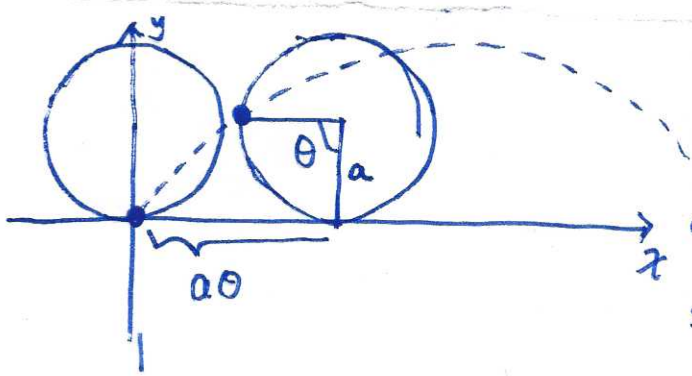
a can be chosen by solving

$$\begin{cases} a[\theta_2 - \sin\theta_2] = x_2 \\ a[1 - \cos\theta_2] = y_2 \end{cases}$$



isochronus curve

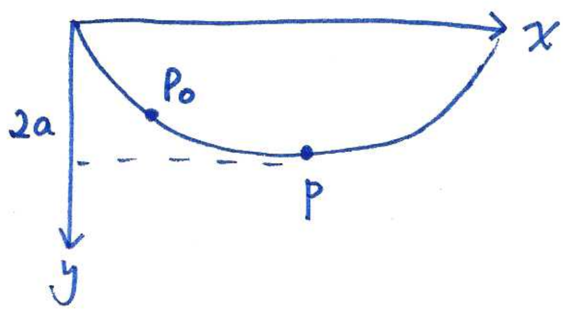
Let us reverse the direction of y-axis



consider the trajectory of a fixed point on the wheel of radius a. The wheel rolls along straight line.

$$\Rightarrow \begin{cases} x = a\theta - a\sin\theta \\ y = a - a\cos\theta \end{cases}$$

For a pendulum, the time from $P_0 \to P$ is independent of the position of P_0



$$dt = \frac{ds}{v} \qquad ds = \left[(dx)^2 + (dy)^2\right]^{1/2} = a\, d\theta\left[(1-\cos\theta)^2 + \sin^2\theta\right]^{1/2}$$

$$= \sqrt{2}\, a\, d\theta\, [1-\cos\theta]^{1/2}$$

$$v = \sqrt{2g\Delta y} = \sqrt{2g}\sqrt{\cos\theta_0 - \cos\theta}$$

$$dt = \sqrt{\frac{a}{g}}\sqrt{\frac{1-\cos\theta}{\cos\theta_0 - \cos\theta}}\, d\theta$$

$$\Rightarrow \text{time}(P_0 \to P) = \sqrt{\frac{a}{g}}\int_{\theta_0}^{\pi}\sqrt{\frac{1-\cos\theta}{\cos\theta_0 - \cos\theta}}\, d\theta .$$

plug in $\theta = \pi - 2\alpha \rightarrow$

For θ from 0 to π,

α takes $\pi/2$ to 0.

$$\sqrt{\frac{1-\cos\theta}{\cos\theta_0 - \cos\theta}} = \left[\frac{2\cos^2\alpha}{\cos\theta_0 + 1 - 2\sin^2\alpha}\right]$$

$$\sqrt{\frac{1-\cos\theta}{\cos\theta_0-\cos\theta}}\, d\theta = \left[\frac{1}{-\sin^2\alpha + \frac{1+\cos\theta_0}{2}}\right]^{1/2} \cos\alpha\,(-2)\, d\alpha$$

$$= -2\left[\frac{1}{\cos^2\frac{\theta_0}{2} - \sin^2\alpha}\right]^{1/2} d\sin\alpha = -2\left[\frac{1}{\sin^2\alpha_0 - \sin^2\alpha}\right]^{1/2} d\sin\alpha$$

Set $u = \sin\alpha$, at $\theta = \theta_0 \Rightarrow \alpha_0 = \frac{\pi}{2} - \frac{\theta}{2}$

$\theta = \pi \Rightarrow \alpha = 0$

$$\Rightarrow \text{time}(P_0 \to P) = \sqrt{a/g} \int_{\alpha_0}^{0} -2\left[\frac{1}{\sin^2\alpha_0 - \sin^2\alpha}\right]^{1/2} d\sin\alpha$$

$$= 2\sqrt{a/g} \int_0^{u_0} \left[\frac{1}{u_0^2 - u^2}\right]^{1/2} du$$

$$= 2\sqrt{a/g} \int_0^1 \left[\frac{1}{1-(u/u_0)^2}\right] d(u/u_0) = 2\sqrt{a/g}\, \arcsin(u/u_0)\Big|_0^{u_0}$$

$$= \pi\sqrt{a/g}$$