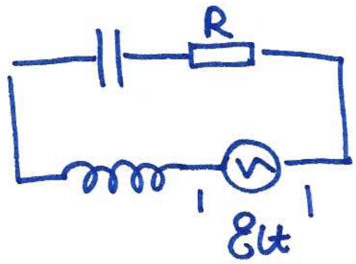
Lect 12 : More on oscillations (II)

§ Driven damped oscillations

In addition to the restoring force, friction, we further consider a driving force $F(t)$, then $m\ddot{x} + b\dot{x} + kx = F(t)$.

or, similarly, in the LC circuit, there's an additional driving EMF



$$\Rightarrow \quad R\dot{q} + \frac{q}{C} = \mathcal{E}(t) - L\ddot{q}$$

Both systems can be described by the same Eq

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t) \quad \text{where } f(t) = \frac{F(t)}{m}.$$

This is a constant coefficient, linear, inhomogeneous ODE.

The solution, due to the superposition principle, can be written as

$$x(t) = x_h(t) + x_p(t).$$

$x_h(t)$ is the general solutions of the homogenous part, satisfying

$$\ddot{x}_h + 2\beta\dot{x}_h + \omega_0^2 x_h = 0.$$

$x_p(t)$ is a perticular solution of the inhomogenous ODE, satisfying $\ddot{x}_p + 2\beta\dot{x}_p + \omega_0 x_p = f(t)$.

Since $x_h(t)$ decays with time, the long term behavior is determined by $x_p(t)$.

* A sinusoidal driving force: $f(t) = f_0 \cos\omega t$

A trick to solve this ODE is to use complex #, define

$$\ddot{y} + 2\beta\dot{y} + \omega_0^2 y = f_0 \sin\omega t, \quad \text{and} \quad z = x + iy$$

$\Rightarrow$ $\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f_0 e^{i\omega t}$. After we solve z, then take its real part.

Try a particular solution

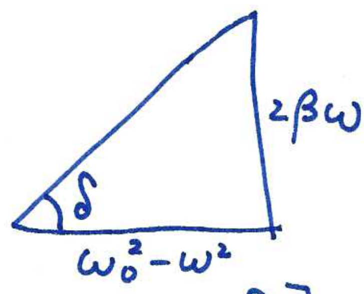
$$z_p(t) = C e^{i\omega t} \Rightarrow C[-\omega^2 + 2i\beta\omega + \omega_0^2] = f_0$$

$$C = \frac{f_0}{\omega_0^2 - \omega^2 + 2i\beta\omega}$$

Let us express $C = A e^{-i\delta}$ $\Rightarrow$ $\boxed{A^2 = \frac{f_0^2}{(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2}}$

& $\delta = \mathrm{Arg}(\omega_0^2 - \omega^2 + 2i\beta\omega)$

$$\boxed{\delta = \arctan\frac{2\beta\omega}{\omega_0^2 - \omega^2}}$$



$\Rightarrow$ $x_p(t) = \mathrm{Re}[A e^{-i\delta + i\omega t}] = A\cos[\omega t - \delta]$

$$\Rightarrow x(t) = A\cos[\omega t - \delta] + \underbrace{c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}}_{\text{transients}} \to 0 \text{ as } t \to \infty$$

For weakly damped systems

$$x(t) = A\cos[\omega t - \delta] + A_{tr}\, e^{-\beta t} \cos[\omega_1 t - \delta_{tr}]$$

where $\omega_1 = \sqrt{\omega^2 - \beta^2}$.

Example: A driven damped linear oscillator released at the origin at time $t = 0$ from the rest with the following parameters: Drive frequency $\omega = 2\pi$, natural frequency $\omega_0 = 5\omega$, decay constant $\beta = \omega_0/20$, and driving amplitude $f_0 = 100$.

Solution: $$x(t) = A\cos[\omega t - \delta] + e^{-\beta t}[B_1 \cos\omega_1 t + B_2 \sin\omega_1 t]$$

$$\rightarrow v(t) = -A\omega \sin(\omega t - \delta] - \beta e^{-\beta t}[B_1 \cos\omega_1 t + B_2 \sin\omega_2 t + e^{-\beta t} \omega_1[-B_1 \sin\omega_1 t + B_2 \cos\omega_1 t]$$

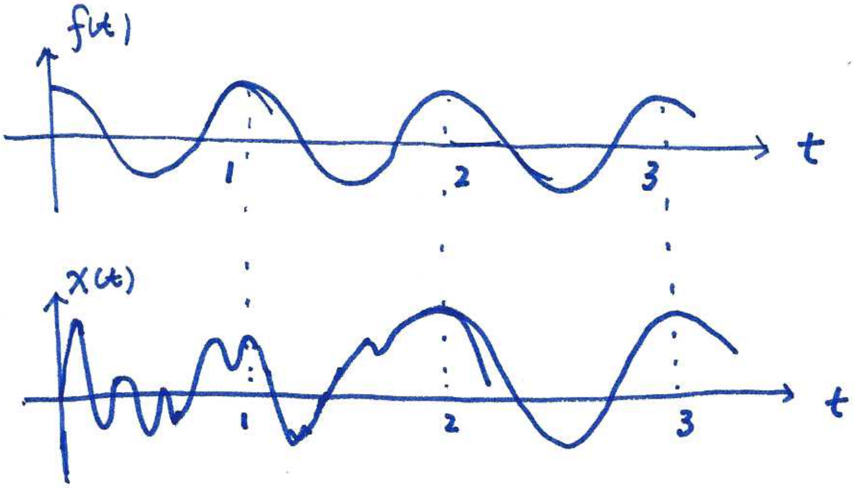
From $$A = \frac{f_0}{[(\omega^2 - \omega_0^2)^2 + 4\beta^2\omega^2]^{1/2}} = 1.06$$

$$\delta = \tan^{-1}\frac{2\beta\omega}{\omega_0^2 - \omega^2} = 0.0208$$

$$\omega_1 = \sqrt{\omega_0^2 - \beta^2} = 9.987\pi.$$

$$\begin{cases} x_0 = A\cos\delta + B_1 \\ v_0 = +A\omega\sin\delta - \beta B_1 + \omega_1 B_2 \end{cases} \Rightarrow \begin{array}{l} B_1 = x_0 - A\cos\delta \\ B_2 = \frac{1}{\omega_1}[v_0 - \omega A \sin\delta + \beta B_1] \end{array}$$

plug in $x_0 = v_0 = 0 \Rightarrow \begin{cases} B_1 = -1.05 \\ B_2 = -0.0572 \end{cases}$



transient motion depends on the initial values x_0 and v_0, but its decays. Different x_0 and v_0 lead to the same stable motion. This motion is called — attractor.

§ resonance

$$Z_p(t) = C\, e^{i\omega t}$$ under the drive force $f = f_0 e^{i\omega t}$

with $$C = \frac{f_0}{\omega_0^2 - \omega^2 + i 2\beta\omega} = A e^{-i\delta}$$

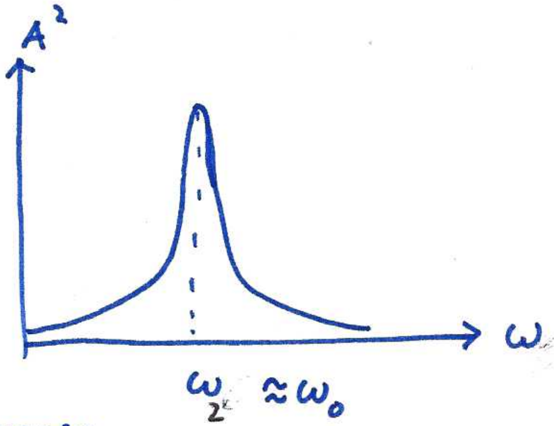
① If ω_0 and ω are very different, then $|C| << f_0$. When $\omega_0 \to \omega$, the amplitude of A reaches the maximum (fix ω).

This is called resonance.

Good resonance — LRC circuit, selecting radio wave frequency

bad resonance — a marching troop can make a bridge collapse

$$A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$



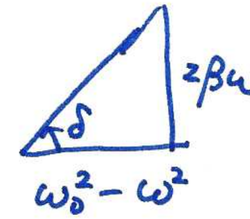
If fix ω_0, the A reaches the maximum at a slightly smaller frequency than ω_0.

$$\frac{\partial}{\partial \omega^2}\left[(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2\right]\Big|_{\omega=\omega_2} = 0$$

$$-2(\omega_0^2-\omega^2) + 4\beta^2 = 0 \Rightarrow \omega_2 = \left[\omega_0^2 - 2\beta^2\right]^{1/2}.$$

Since $\omega_2 \simeq \omega_0$ in the case of $\beta \ll \omega_0$, we often donot distinguish ω_2 and ω_0.

☆ Let's look at the phase difference $\delta = \arctan \dfrac{2\beta\omega}{\omega_0^2 - \omega^2}$

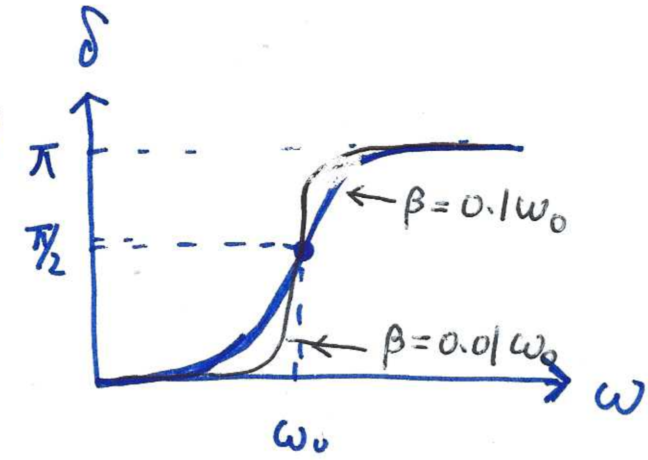


① at $\omega \ll \omega_0$, $\delta \to 0$

② at $\omega \gg \omega_0$ $\delta \to \pi$

} far from resonance "C" is real

→ The osscillater cannot follow – has "π"-phase difference!



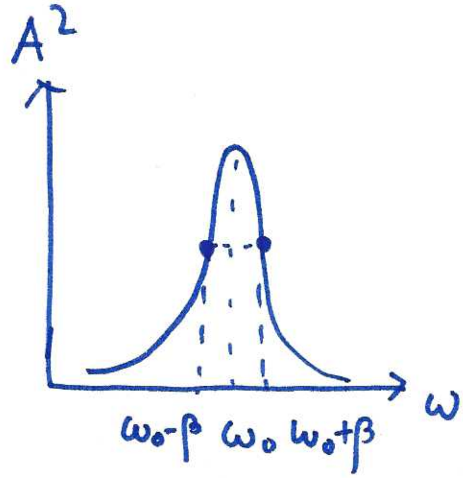
✱ Width of the resonance

At $\beta \ll \omega_0$, A^2 reaches half maximum at $(\omega_0^2 - \omega^2)^2 = 4\beta^2\omega_0^2$

$$\Rightarrow \quad \omega_0^2 - \omega^2 = \pm 2\beta\omega_0 \quad \Rightarrow \omega^2 = \omega_0^2 \pm 2\beta\omega_0 \approx (\omega_0 \pm \beta)^2$$

or $\omega \approx \omega_0 \pm \beta$. $\rightarrow$ Define FWHM $\approx 2\beta$

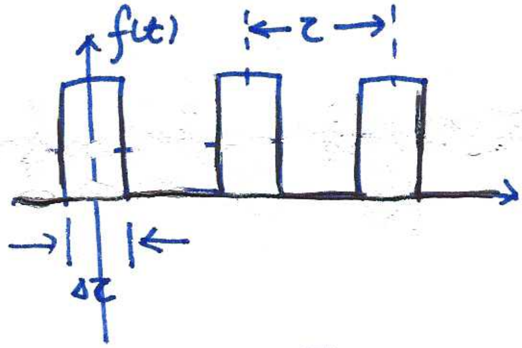
or HWHM $\approx \beta$.



§ General driving force — Fourier analysis

Consider a periodic driving force $f(t+\tau) = f(t)$.

$$f(t) = \sum_{n=0}^{\infty} \left[a_n \cos n\omega t + b_n \sin n\omega t \right]$$



with $\omega = \dfrac{2\pi}{\tau}$

$$a_n = \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} f(t) \cos n\omega t \, dt$$

$(n \geq 1)$.

$a_0 = \bar{f}$, we often set $\bar{f} = 0$.

$$b_n = \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} f(t) \sin n\omega t \, dt$$

$$a_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\cos n\omega t\, dt = \frac{2 f_{max}}{\tau}\int_{-\Delta\tau/2}^{\Delta\tau/2} \cos n\omega t\, dt$$

$$= \frac{4 f_{max}}{\tau}\int_0^{\Delta\tau/2} \cos\left(\frac{2\pi n t}{\tau}\right) dt = \frac{2 f_{max}}{\pi n}\sin\left(\frac{\pi n \Delta\tau}{\tau}\right).$$

- Suppose

$$f(t) = \sum_{n=0}^{\infty} f_n \cos(n\omega t - \delta_n') \rightarrow \sum_{n=0}^{\infty} f_n' e^{in\omega t}$$

phase δ_n' can be absorbed in f_n', which is complex

$$X_n(t) = C_n e^{in\omega t}$$

where $$C_n = \frac{f_n'}{\omega_0^2 - (n\omega)^2 + 2i\beta\omega n} = A_n e^{-i(\delta_n + \delta_n')}$$

$$\Rightarrow A_n = \frac{|f_n'|}{\left[(\omega_0^2 - n^2\omega^2)^2 + 4\beta^2 n^2\omega^2\right]^{1/2}}$$

$$\delta_n = \tan^{-1}\frac{2\beta n\omega}{\omega_0^2 - n^2\omega^2}$$

- or in terms of real numbers, $X(t) = \sum_{n=0}^{\infty} A_n \cos[n\omega t - \delta_n - \delta_n']$.

§ Root - Mean - Square (RMS) —— Parseval's theorem

$$X_{rms} = \sqrt{\langle x^2 \rangle} \quad , \quad \langle x^2 \rangle = \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} x^2 \, dt$$

if $x(t) = \sum_{n=0}^{\infty} A_n \cos(n\omega t - \delta_n) \leftarrow$ orthogonal basis

then $\langle x^2 \rangle = A_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} A_n^2 \xrightarrow{\text{c.f}} r^2 = x^2 + y^2 + z^2$

$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

plug in the expression of A_n, we have

$\langle x^2 \rangle$ reaches maxima at the driving period τ satisfying

$$\tau = n\tau_0 \quad , \quad \text{or} \quad \omega = \frac{1}{n}\omega_0$$

$\uparrow$ driving period $\quad \uparrow$ natural period