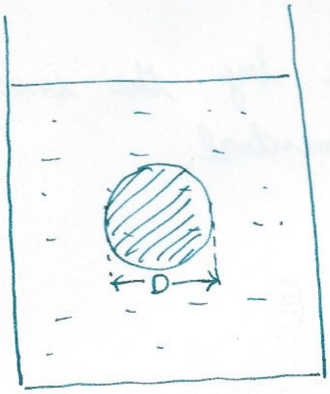
PHYS 110A

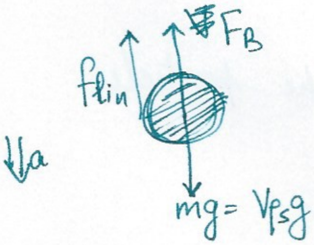
HW II

Taylor 2.10)

5



$\downarrow \hat{y}$



D: diameter of ball

ρ_s : density of ball

V: volume of ball

ρ_g: density of glycerine

η: viscosity of glycerine

F_B: Buoyant force on the ball

f_{lin}: linear drag on ball

$v(t)$: velocity of ball at time t

V_∞ : terminal velocity of ball

m: mass of ball

From Newton's II law,

$$m\frac{dv}{dt} = mg - F_B - f_{lin} \longrightarrow ①$$

From Archimedes principle, $F_B = V\rho_g g$.

We know that $f_{lin} = 3\pi\eta D\, v(t)$

Now, $m = V\rho_s$ (by definition)

Plugging this into ①,

$$\frac{dv}{dt} = g\left(1 - \frac{\rho_g}{\rho_s}\right) - \frac{3\pi\eta D}{m} v$$

We can solve this by separating variables,

$$\int_0^v \frac{dv}{g\left(1-\frac{\rho_g}{\rho_s}\right) - \frac{3\pi\eta D}{m} v} = \int_0^t dt$$

$$\ln\left(1 - \frac{\frac{3\pi\eta D}{m} v}{g\left(1-\frac{\rho_g}{\rho_s}\right)}\right) = -\frac{3\pi\eta D}{m} t$$

$$\Rightarrow \boxed{v(t) = \frac{mg}{3\pi\eta D}\left(1 - \frac{\rho_g}{\rho_s}\right)\left[1 - \exp\left(-\frac{3\pi\eta D}{m}t\right)\right]} \rightarrow (2)$$

The characteristic time τ is given by the inverse of the coefficient of t in the exponential.

So, $$\boxed{\tau = \frac{m}{3\pi\eta D}} \longrightarrow (3)$$

The terminal velocity is the velocity attained by the particle after a long time (which is physicist speak for $t \to \infty$)

$$\boxed{V_\infty = V(t \to \infty) = \frac{mg}{3\pi\eta D}\left(1 - \frac{\rho_g}{\rho_s}\right)} \longrightarrow (4)$$

N.B: The addition of the ~~bo~~ buoyant force does not affect τ but reduces the terminal velocity by a factor ρ_g/ρ_s.

Using (3) & (4), we can rewrite eqn (2) as

$$v(t) = V_\infty\left(1 - e^{-t/\tau}\right)$$

Say the ball reaches 95% of terminal velocity at time t^*. Then,

$$0.95\,V_\infty = V_\infty\left(1 - e^{-t^*/\tau}\right)$$

$$\Rightarrow \boxed{t^* = \tau \ln 20}$$

(b) $f_{lin} = 3\pi\eta D v_\infty$

$f_{quad} = \frac{\pi k}{4} \rho_g \frac{D v_\infty}{\eta}$

$$\Rightarrow \boxed{\frac{f_{quad}}{f_{lin}} = \frac{k}{12} \frac{\rho_g D v_\infty}{\eta}}$$

You can plug ~~in~~ the numbers for density, viscosity etc, to get

$\tau = 1.4 \times 10^{-4}$ s $\qquad t^* \simeq 4.2 \times 10^{-4}$ s

$v_\infty = 0.11$ cm/s

$\frac{f_{quad}}{f_{lin}} \sim 5 \times 10^{-6}$

$\because$ $f_{quad} \sim 10^{-6} \times f_{lin}$, it is a reasonable approximation to neglect f_{quad}.

(Taylor 2.11)

5

Say the mass of the object is m & the drag coefficient is b.

Then, from Newton's II law,

$$m\frac{dv_y(t)}{dt} = -mg - bv_y(t)$$, where $v_y(t)$ is the velocity of the object at time t.

$$\Rightarrow \quad \frac{dv_y}{g + b/m\, v_y} = -dt$$

Integrating with initial velocity v_0, we get

$$\int_{v_0}^{v_y} \frac{dv_y}{g + b/m\, v_y} = -\int_0^t dt$$

$$\Rightarrow \frac{m}{b} \ln\left(\frac{g + b/m\, v_y}{g + b/m\, v_0}\right) = -t$$

$$\Rightarrow \quad \boxed{v_y(t) = \frac{-mg}{b} + \left(\frac{mg}{b} + v_0\right) e^{-tb/m}}$$

By definition,

$$\cancel{dy} \; v_y(t) = \frac{dy}{dt}$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{-mg}{b} + \left(\frac{mg}{b} + v_0\right) e^{-tb/m}$$

Integrating with initial ~~y~~ position at the origin

$$\int_0^{y(t)} dy = \int_0^t \left[\frac{-mg}{b} + \left(\frac{mg}{b} + v_0\right) e^{-tb/m}\right] dt$$

$$\Rightarrow \quad y(t) = -\frac{mg}{b}t + \frac{m}{b}\left(\frac{mg}{b} + v_0\right)\left(1 - e^{-tb/m}\right)$$

(b) At the highest point, the velocity of the ~~ball~~ object would be 0. Say, the corresponding time is t^*. Then,

$$v_y(t^*) = 0$$

$$\Rightarrow \quad \frac{mg}{b} = \left(\frac{mg}{b} + v_0\right) e^{-t^* b/m}$$

$$\Rightarrow \quad \boxed{t^* = \frac{m}{b}\ln\left(1 + \frac{v_0 b}{mg}\right)}$$

The corresponding position, $y_{max} = y(t^*)$

$$y_{max} = y(t^*) = -\left(\frac{m}{b}\right)^2 g \ln\left(1 + \frac{v_0 b}{mg}\right) - \frac{m}{b}\left(\frac{mg}{b} + v_0\right)\left(-1 + \frac{1}{1 + v_0 b/mg}\right)$$

$$\Rightarrow \quad y_{max} = \frac{m}{b}\left(\frac{mg + bv_0}{b}\right)\left(\frac{v_0 b}{mg}\right)\frac{mg}{mg + bv_0} - \left(\frac{m}{b}\right)^2 g \ln\left(1 + \frac{v_0 b}{mg}\right)$$

$$\Rightarrow \quad \boxed{y_{max} = \frac{v_0 m}{b} - \left(\frac{m}{b}\right)^2 g \ln\left(1 + \frac{v_0 b}{mg}\right)}$$

(c) If ~~b is small~~ $b \to 0$, $\frac{v_0 b}{mg} << 1$. Then,

$$\ln\left(1 + \frac{v_0 b}{mg}\right) \approx \left(\frac{v_0 b}{mg}\right) - \frac{1}{2}\left(\frac{v_0 b}{mg}\right)^2$$

$$\Rightarrow \quad y_{max} \approx \frac{v_0 m}{b} - \frac{m^2}{b^2} g\left(\frac{v_0 b}{mg} - \frac{1}{2}\frac{v_0^2 b^2}{m^2 g^2}\right)$$

$$y_{max} = \frac{1}{2}\frac{v_0^2}{g}$$, which is the result for an object in vacuum.

(Taylor 2.12) "The $v\,dv/dx$" rule:

3

$$\dot{v} = \frac{dv}{dt}$$

mulibply & divide by dx,

$$\dot{v} = \frac{dv}{dt} = \frac{dx}{dx}\frac{dv}{dt} = \left(\frac{dx}{dt}\right)\left(\frac{dv}{dx}\right)$$

Now $\frac{dx}{dt} = v$

$$\Rightarrow \boxed{\dot{v} = v\frac{dv}{dx}}$$

$$v\frac{dv}{dx} = \frac{1}{2}\frac{d(v^2)}{dx} \Rightarrow \boxed{\dot{v} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{1}{2}\frac{d(v^2)}{dx}}$$

Note that this proof is not strictly "proper" in the mathematical sense (in particular the $\frac{dx}{dx}\frac{dv}{dt} = \frac{dx}{dt}\frac{dv}{dx}$ step)

Now, from Newton's II law,

$$m\frac{dv}{dt} = F(x)$$

$$\Rightarrow \frac{dv}{dt} = \frac{F(x)}{m} \Rightarrow \frac{1}{2}\frac{d(v^2)}{dx} = \frac{F(x)}{m}$$

$$\Rightarrow d(v^2) = \frac{2}{m}F(x)\,dx$$

Integrate on both sides with initial velocity v_0 at position x_0

$$\Rightarrow \int_{v_0}^{v} d(v^2) = \frac{2}{m}\int_{x_0}^{x} F(x')\,dx'$$

(The change from x to x' is purely for notational clarity & nothing more)

$$\Rightarrow \quad \boxed{v^2 - v_0^2 = \frac{2}{m}\int_{x_0}^{x} F(x')\,dx'}$$

N.B: Rearranging the equation gives,

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = \int_{x_0}^{x} F(x')\,dx'.$$

The LHS is the difference between the final kinetic energy & initial kinetic energy of the particle.
The RHS is the work done by the forces acting on the particle.

Therefore, we have essentially proved that the change in kinetic energy of ~~to~~ a particle is equal to the work done on it, ie, conservation of energy (in some form).

This exercise shows us how Newton's law's lead to conservation of energy.

Note also, that we could do the derivation in reverse, starting from $\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = \int_{x_0}^{x} F(x')\,dx'$ to derive Newton's II law.

(Taylor 2.20)

Start with eq 2.37 in chapter 2,

$$y = \frac{v_{y0} + v_{ter}}{v_{x0}} x + v_{ter}\tau \ln\left(1 - \frac{x}{v_{x0}\tau}\right)$$

$\because$ the projectile is thrown at 45°, $v_{y0} = v_{x0} = v_0$ (say)

$\tau = \frac{v_{ter}}{g}$ (eq 2.34 in chapter 2)

$$\Rightarrow \quad y = \frac{v_0 + v_{ter}}{v_0} x + \frac{v_{ter}^2}{g} \ln\left(1 - \frac{xg}{v_0 v_{ter}}\right)$$

Now, for any physical system, one can find (generally), a scale for length & time that is "most convenient" for that system. In this case for ~~exaumgle~~ example, the velocity v_0 & g set scales for length & time. setting these to be 1,

$$y = (1 + v_{ter})x + v_{ter}^2 \ln\left(1 - \frac{x}{v_{ter}}\right)$$