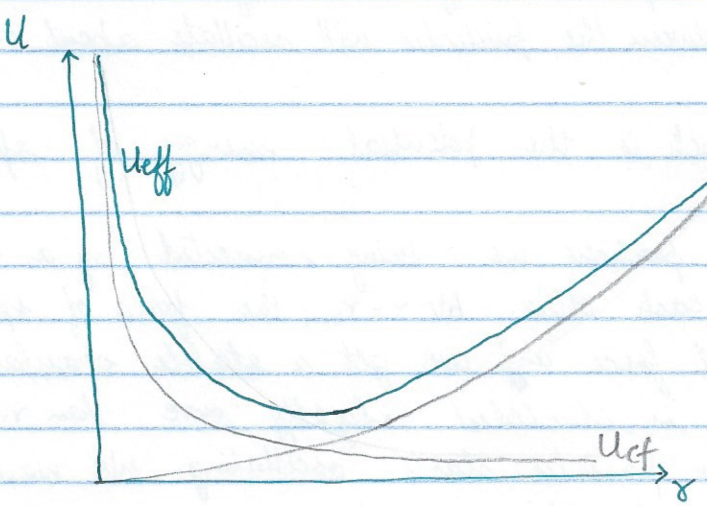
PHYS 110A HW #5

Taylor 8.13 (a) $U(r) = \frac{1}{2}kr^2$ (Given)

$U_{cf}(r) = \frac{\ell^2}{2\mu r^2}$ (by definition) [ℓ is angular momentum]

$U_{eff}(r) = U(r) + U_{cf}(r)$

$= \frac{1}{2}kr^2 + \ell^2/2\mu r^2$



(b) At the equilibrium separation, the particles will circle each other with constant $r (= r_0)$. The condition for finding r_0 is that

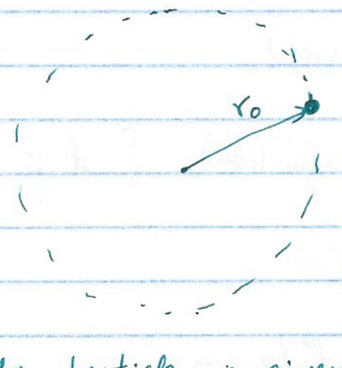
$$\left.\frac{dU_{eff}}{dr}\right|_{r=r_0} = 0$$

$$\frac{dU_{eff}}{dr} = kr - \frac{\ell^2}{\mu r^3}$$

At $r = r_0$,

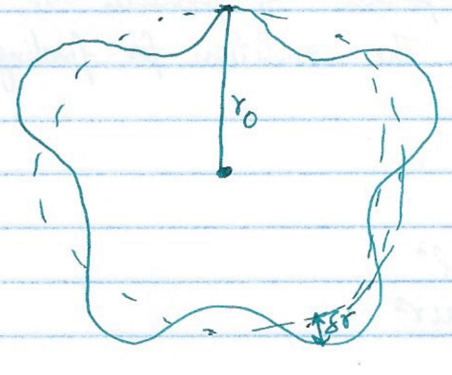
$$\frac{dU_{eff}}{dr} = 0, \Rightarrow kr_0 - \frac{\ell^2}{\mu r_0^3} = 0 \Rightarrow \boxed{r_0 = \left(\frac{\ell^2}{\mu k}\right)^{1/4}}$$

c At $r = r_0$, the particles will go around each other in a circle of radius r_0.



However, if the particle is given a slight kick (radially outward, let's say), The distance between the particles will oscillate about $r = r_0$.

Note: $U = \frac{1}{2} k r^2$ which is the potential energy of spring. So, you can think of the particles as being connected by a spring and going around each other. At $r = r_0$, the force of spring balances the centrifugal force and we get a stable circular orbit.
If the spring is stretched slightly more than $r = r_0$, the distance between the particles starts oscillating. We want to find the frequency of this oscillation.



If r is the distance between the particles, we define $\delta r = r - r_0$ as the deviation of the distance from the equilibrium distance r_0.

Then

$$U_{eff}(r) = U_{eff}(r_0 + \delta r)$$

$$= U_{eff}(r_0) + \delta r \left(\frac{dU_{eff}}{dr}\bigg|_{r=r_0}\right) + \frac{1}{2}(\delta r)^2 \left(\frac{d^2 U_{eff}}{dr^2}\right)_{r=r_0} + \text{higher order terms}$$

$$\approx U_{eff}(r_0) + \delta r \left(\frac{dU_{eff}}{dr}\right)_{r=r_0} + \frac{1}{2}(\delta r)^2 \left(\frac{d^2 U_{eff}}{dr^2}\right)_{r=r_0}$$

Note that $\left(\frac{dU_{eff}}{dr}\right)_{r=r_0} = 0$ (That's how we found r_0)

$$\Rightarrow U_{eff}(r) \approx \frac{1}{2} k r_0^2 + \frac{\ell^2}{2\mu r_0^2} + \frac{1}{2}(\delta r)^2 \left(\frac{d^2 U_{eff}}{dr^2}\right)_{r=r_0}$$

$$\frac{d^2 U_{eff}}{dr^2} = k + \frac{3\ell^2}{\mu r^4}$$

$$\Rightarrow U_{eff}(r) \approx \frac{1}{2} k r_0^2 + \frac{\ell^2}{2\mu r_0^2} + \frac{1}{2}(\delta r)^2 \left(k + \frac{3\ell^2}{\mu r_0^4}\right)$$

$$= \frac{\mu k r_0^4 + \ell^2}{2\mu r_0^2} + \frac{1}{2}(\delta r)^2 \left[k + \frac{3\ell^2}{\mu r_0^4}\right]$$

$$r_0 = \left(\frac{\ell^2}{\mu k}\right)^{1/4}$$

$$U_{eff}(r) = \frac{\ell^2 (\mu k)^{1/2}}{\mu (\ell^2)^{1/2}} + \frac{1}{2}(\delta r)^2 (3k)$$

$$\delta r = r - r_0$$

$$\Rightarrow U_{eff}(r) = \ell \sqrt{k/\mu} + \frac{3}{2} k (r - r_0)^2$$

$$U_{eff}(r) = \quad l\sqrt{\frac{k}{\mu}} \quad + \quad \frac{3k}{2}(r-r_0)^2$$

Now, the equation of motion for r is

$$\mu\ddot{r} = F$$

but by definition, $F = -\dfrac{dU_{eff}}{dr}$

$$\Rightarrow \quad \mu\ddot{r} = -3k(r-r_0)$$

$$\Rightarrow \quad \boxed{\ddot{r} = -3\frac{k}{\mu}(r-r_0)}$$

Not that this is identical to the equation for a simple harmonic oscillator with equilibrium position r_0 & frequency $\omega = \sqrt{\dfrac{3k}{\mu}}$

$$\boxed{\omega = \sqrt{\frac{3k}{\mu}}}$$

This is the frequency of the small oscillations about r_0

Taylor (8.17) Define

$$G = \vec{r}.\vec{p}$$

$$\frac{dG}{dt} = \left(\frac{d\vec{r}}{dt}\right).\vec{p} + \vec{r}.\left(\frac{d\vec{p}}{dt}\right)$$

$$\frac{d\vec{r}}{dt} = \vec{v} \; . \quad \vec{p} = m\vec{v} \quad \text{(by definition)}$$

$$\frac{d\vec{p}}{dt} = \vec{F} \quad \text{(Newton's II law)}$$

$$\Rightarrow \frac{dG}{dt} = m\vec{v}.\vec{v} + \vec{r}.\vec{F}$$

$$\frac{dG}{dt} = 2\left(\frac{1}{2}mv^2\right) + \vec{r}.\vec{F}$$

$$\Rightarrow \frac{dG}{dt} = 2T + \vec{r}.\vec{F}$$, where T is the kinetic energy of the system.

$$\Rightarrow \int_{G(0)}^{G(t)} dG = \int_0^t \left(2T + \vec{r}.\vec{F}\right) dt$$

Now, note that time average of a quantity $A(t)$ is defined as

$$\langle A \rangle = \frac{1}{t}\int_0^t A(t')\, dt'$$

With this definition,

$$\int_0^t \left(2T + \vec{r}.\vec{F}\right) dt = \left[2\langle T\rangle + \langle \vec{r}.\vec{F}\rangle\right] \times t$$

$$\Rightarrow G(t) - G(0) = \left[2\langle T\rangle + \langle \vec{r}.\vec{F}\rangle\right] \times t$$

$$\Rightarrow \boxed{\frac{G(t) - G(0)}{t} = 2\langle T\rangle + \langle \vec{r}.\vec{F}\rangle}$$

(b) $G = \vec{r} \cdot \vec{p}$

If the particle is in a bound orbit $\vec{r}$ & $\vec{p}$ are always finite. $\Rightarrow$ G is always finite

$\Rightarrow$ $G(t) - G(0)$ is finite for all t.

$\Rightarrow$ For t large $(t \to \infty)$, $\frac{G(t) - G(0)}{t} \longrightarrow 0$.

(c) Given $\vec{F}$ comes from the potential $U = kr^n$.

$$\Rightarrow \vec{F} = -\frac{\partial U}{\partial \vec{r}} = -knr^{n-2}\vec{r}$$

$\Rightarrow$ From parts (a) & (b), we have

$$2\langle T \rangle + \langle \vec{F} \cdot \vec{r} \rangle = 0$$

$$\Rightarrow \quad 2\langle T \rangle + \langle -knr^{n-2}\vec{r} \cdot \vec{r} \rangle = 0$$

$$\Rightarrow \quad 2\langle T \rangle - n\langle kr^n \rangle = 0$$

but $\langle kr^n \rangle = \langle U \rangle$

$$\Rightarrow \quad \boxed{\langle T \rangle = n/2 \langle U \rangle}$$

Taylor (8.23) From Newton's II law,

$$m\ddot{r} = F(r) + F_{centrifugal}(r)$$

$$\Rightarrow \quad m\ddot{r} = -\frac{k}{r^2} + \frac{\lambda}{r^3} + \frac{l^2}{mr^3}$$ where $l = mr^2 \frac{d\phi}{dt}$ is the angular momentum.

$$\boxed{m\ddot{r} = -\frac{k}{r^2} + \frac{\lambda}{r^3} + \frac{l^2}{mr^3}} \longrightarrow (1)$$

Let $r = 1/u$.

And $dt = \frac{mr^2 d\phi}{l}$ $\left(\because l = mr^2 \frac{d\phi}{dt} \right)$

$$dt = \frac{mr^2}{l} d\phi$$

Then

$$\frac{dr}{dt} = \frac{d}{dt}\left(\frac{1}{u}\right) = \frac{-1}{u^2} \frac{du}{dt}$$

$$= -\frac{1}{u^2} \frac{l\,du}{mr^2\, d\phi}$$

$$= \frac{-1}{u^2} \frac{l}{m} u^2 \frac{du}{d\phi} = -\frac{l}{m} \frac{du}{d\phi}$$

$$\frac{d^2r}{dt^2} = \frac{d}{dt}\left(\frac{dr}{dt}\right) = \frac{d}{dt}\left(-\frac{l}{m}\frac{du}{d\phi}\right) = -\frac{l}{m} \frac{d}{dt}\left(\frac{du}{d\phi}\right)$$

$$= -\frac{l}{m} \frac{l\,d}{mr^2 d\phi}\left(\frac{du}{d\phi}\right) = \frac{-l^2 u^2}{m^2} \frac{d^2u}{d\phi^2}$$

Plugging this into (1),

$$-\frac{l^2 u^2}{m} \frac{d^2u}{d\phi^2} = -ku^2 + \left(\lambda + \frac{l^2}{m}\right) u^3$$

$$\Rightarrow \boxed{\frac{d^2u}{d\phi^2} = \frac{mk}{l^2} - \left(\frac{m\lambda + l^2}{l^2}\right) u}$$

This equation is similar to that of a harmonic oscillator. So, we try the solution

$$u(\phi) = A\cos(\beta\phi) + B$$

Plugging this in, we get,

$$-A\beta^2\cos(\beta\phi) = \frac{mk}{l^2} - \left(\frac{m\lambda}{l^2} + 1\right)(A\cos\beta\phi + B)$$

~~Comparing co~~

Equating coefficients of powers of $\cos\beta\phi$, we get,

$$-A\beta^2 = -\left(\frac{m\lambda}{l^2} + 1\right) A \quad \text{and} \quad 0 = \frac{mk}{l^2} - B\left(\frac{m\lambda}{l^2} + 1\right)$$

$$\Rightarrow \boxed{\beta = \sqrt{\frac{m\lambda}{l^2} + 1}} \quad \boxed{B = \frac{mk/l^2}{1 + m\lambda/l^2}}$$

$$\Rightarrow u(\phi) = A\cos\beta\phi + B$$

$$\Rightarrow r(\phi) = \frac{1}{B + A\cos\beta\phi}$$

$$\boxed{r(\phi) = \frac{1/B}{1 + A/B\cos\beta\phi}}, \quad \beta = \sqrt{\frac{1 + m\lambda}{l^2}}, \quad B = \frac{mk/l^2}{1 + m\lambda/l^2}$$

Comparing this to the form of the solution in the text,

$$c = 1/B = \frac{1 + m\lambda/l^2}{mk/l^2}, \quad \epsilon = \frac{A}{B} = \frac{Al^2}{mk}\left[1 + \frac{m\lambda}{l^2}\right], \quad \beta = \sqrt{\frac{1 + m\lambda}{l^2}}$$

So, $$r(\phi) = \frac{c}{1+\epsilon\cos\beta\phi}$$

For $0<\epsilon<1$, $1+\epsilon\cos\beta\phi > 0$ for all β & ϕ.

$\Rightarrow$ $r(\phi)$ is bounded. So, the orbit will be bounded for $0<\epsilon<1$.

(c) The ~~orbits~~ orbits are closed if

$$r(\phi) = r(\phi + 2\pi)$$
$$\Rightarrow \cos(\beta\phi) = \cos(\beta\phi + \beta 2\pi)$$

$\Rightarrow$ If β is an integer, the ~~orbit~~ orbit is closed.

If $\lambda \to 0$, $\beta \to 1$ & you recover the usual kepler orbits.