PHYS 100B (Prof. Congjun Wu)
Solution to HW 3

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Problem 1 (Griffiths 5.24)

If $\mathbf{B}$ is uniform, show that $\mathbf{A}(\mathbf{r}) = -\frac{1}{2}(\mathbf{r}\times\mathbf{B})$ works. That is, check that $\nabla\cdot\mathbf{A}=0$ and $\nabla\times\mathbf{A}=\mathbf{B}$. Is this result unique, or are there other functions with the same divergence and curl?

Solution: Since $\mathbf{B}$ is uniform, $\nabla\times\mathbf{B}=\mathbf{0}, (\mathbf{r}\cdot\nabla)\mathbf{B}=\mathbf{0}$. And $\nabla\times\mathbf{r}=0, \nabla\cdot\mathbf{r}=3$, we have

$$\begin{aligned}
\nabla\cdot\mathbf{A} &= -\frac{1}{2}\nabla\cdot(\mathbf{r}\times\mathbf{B}) = -\frac{1}{2}\left(\mathbf{B}\cdot(\nabla\times\mathbf{r}) - \mathbf{r}\cdot(\nabla\times\mathbf{B})\right) = 0 \\
\nabla\times\mathbf{A} &= -\frac{1}{2}\nabla\times(\mathbf{r}\times\mathbf{B}) = -\frac{1}{2}\left(\mathbf{r}\left(\nabla\cdot\mathbf{B}\right) + \left(\mathbf{B}\cdot\nabla\right)\mathbf{r} - \mathbf{B}(\nabla\cdot\mathbf{r}) - (\mathbf{r}\cdot\nabla)\mathbf{B}\right) \\
&= -\frac{1}{2}\left(\mathbf{0}+\mathbf{B}-3\mathbf{B}-0\right) = \mathbf{B}.
\end{aligned}$$

Take

$$\mathbf{A}' = \mathbf{A} + \nabla\varphi,$$

$\Rightarrow$

$$\begin{aligned}
\nabla\cdot\mathbf{A}' &= \nabla\cdot\mathbf{A} + \nabla^2\varphi, \\
\nabla\times\mathbf{A}' &= \nabla\times\mathbf{A}.
\end{aligned}$$

So we need φ to be linear in x, y and z so that $\nabla^2\varphi = \left(\partial_x^2+\partial_y^2+\partial_z^2\right)\varphi = 0$. For example, take $\varphi = xy$, $\nabla\varphi = ye_x + xe_y$, $\nabla^2\varphi = 0$.

Problem 2 (Griffiths 5.29)

Use the results of Ex. 5.11 to find the field inside a uniformly charged sphere of total charge Q and radius R, which is rotating at a constant angular velocity ω.

Solution: In Ex. 5.11, we found the vector potential inside a uniformed charged shell with radius R' as Eq. 5.67,

$$\mathbf{A}\left(r,\theta,\varphi\right) = \begin{cases} \frac{\mu_0 R'\omega\sigma}{3} r\sin\theta\hat{\phi}, (r\le R) \\ \frac{\mu_0 R'^4\omega\sigma}{3}\frac{1}{r^2}\sin\theta\hat{\phi}, (r\ge R) \end{cases}.$$

Here, a uniformly charged sphere can be thought as layers of spheres, larger one containing smaller ones inside. The field inside a uniformly charged sphere can be found by integration over R,

$$\begin{aligned}
\mathbf{A}\left(r,\theta,\varphi\right) &= \frac{\mu_0\omega\rho}{3} r\sin\theta\hat{\phi}\int_r^R R'dR' + \frac{\mu_0\omega\rho}{3}\frac{1}{r^2}\sin\theta\hat{\phi}\int_0^r R'^4 dR \\
&= \frac{\mu_0\omega\rho}{3} r\sin\theta\hat{\phi}\frac{1}{2}\left(R^2-r^2\right) + \frac{\mu_0\omega\rho}{3}\frac{1}{r^2}\sin\theta\hat{\phi}\frac{1}{5}r^5 \\
&= \frac{\mu_0\omega\rho}{2} r\sin\theta\left(\frac{1}{3}R^2 - \frac{1}{5}r^2\right)\hat{\phi}.
\end{aligned}$$

In 3D spherical coordinates, the metric is

$$\eta = \begin{pmatrix} h_r & 0 & 0 \\ 0 & h_\theta & 0 \\ 0 & 0 & h_\varphi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r\sin\theta \end{pmatrix},$$

$$\begin{aligned}
\mathbf{B}(r,\theta,\varphi) &= \nabla\times\mathbf{A}(r,\theta,\varphi)\\
&= \frac{1}{h_\theta h_\varphi}\left[\frac{\partial}{\partial\theta}(A_\varphi h_\varphi)-\frac{\partial}{\partial\varphi}(A_\theta h_\theta)\right]\hat{\mathbf{r}}+\frac{1}{h_\varphi h_r}\left[\frac{\partial}{\partial\varphi}(A_r h_r)-\frac{\partial}{\partial r}(A_\varphi h_\varphi)\right]\hat{\theta}+\frac{1}{h_r h_\theta}\left[\frac{\partial}{\partial\theta}(A_r h_r)-\frac{\partial}{\partial r}(A_\theta h_\theta)\right]\hat{\phi}\\
&= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\varphi r\sin\theta)\right]\hat{\mathbf{r}}+\frac{1}{r\sin\theta}\left[-\frac{\partial}{\partial r}(A_\varphi r\sin\theta)\right]\hat{\theta}\\
&= \frac{\mu_0\omega}{2}\frac{Q}{\frac{4}{3}\pi R^3}\left[\frac{1}{\sin\theta}\left(\frac{1}{3}R^2-\frac{1}{5}r^2\right)\frac{\partial}{\partial\theta}\sin^2\theta\hat{\mathbf{r}}-\frac{1}{r}\sin\theta\frac{\partial}{\partial r}\left(\frac{1}{3}R^2r^2-\frac{1}{5}r^4\right)\hat{\theta}\right]\\
&= \frac{\mu_0\omega Q}{4\pi R}\left[\cos\theta\left(1-\frac{3}{5}\frac{r^2}{R^2}\right)\hat{\mathbf{r}}-\sin\theta\left(1-\frac{6}{5}\frac{r^2}{R^2}\right)\hat{\theta}\right].
\end{aligned}$$

Problem 3 (Griffiths 5.30)

(a) Complete the proof of Theorem 2, Sect. 1.6.2. That is, show that any divergenceless vector field $\mathbf{F}$ can be written as the curl of a vector potential $\mathbf{A}$. What you have to do is find A_x, A_y and A_z such that: (i) $\partial A_z/\partial y - \partial A_y/\partial z = F_x$; (ii) $\partial A_x/\partial z - \partial A_z/\partial x = F_y$; and (iii) $\partial A_y/\partial x - \partial A_x/\partial y = F_z$. Here's one way to do it: Pick $A_x = 0$, and solve (ii) and (iii) for A_y and A_z. Note that the "constants of integration" here are themselves functions of y and z—they're constant only with respect to x. Now plug these expressions into (i), and use the fact that $\nabla\cdot\mathbf{F}=0$ to obtain

$$A_y=\int_0^x F_z(x',y,z)\,dx';\,A_z=\int_0^y F_x(0,y',z)\,dy'-\int_0^x F_y(x',y,z)\,dx'.$$

Solution: Pick $A_x = 0$,

$$\begin{aligned}
-\partial A_z/\partial x &= F_y \Rightarrow A_z=-\int_0^x F_y(x',y,z)\,dx'+C_1(y,z),\\
\partial A_y/\partial x &= F_z. \Rightarrow A_y=\int_0^x F_z(x',y,z)\,dx'+C_2(y,z).
\end{aligned}$$

Now plug these expressions into (i),

$$\frac{\partial}{\partial y}\left[-\int_0^x F_y(x',y,z)\,dx'+C_1(y,z)\right]-\frac{\partial}{\partial z}\left[\int_0^x F_z(x',y,z)\,dx'+C_2(y,z)\right]=F_x,$$

$$-\int_0^x\left(\frac{\partial}{\partial y}F_y(x',y,z)+\frac{\partial}{\partial z}F_z(x',y,z)\right)dx'+\frac{\partial}{\partial y}C_1(y,z)-\frac{\partial}{\partial z}C_2(y,z)=F_x,$$

and use the fact that $\nabla\cdot\mathbf{F}=0\Rightarrow$

$$\int_0^x\frac{\partial}{\partial x}F_x(x',y,z)\,dx'+\frac{\partial}{\partial y}C_1(y,z)-\frac{\partial}{\partial z}C_2(y,z)=F_x,$$

$\Rightarrow$

$$\frac{\partial}{\partial y}C_1(y,z)-\frac{\partial}{\partial z}C_2(y,z)=F_x(0,y,z).$$

Take $C_2(y,z)=0$,

$$\begin{aligned}
A_y &= \int_0^x F_z(x',y,z)\,dx',\\
C_1(y,z) &= \int_0^y F_x(0,y',z)\,dy',
\end{aligned}$$

$$\begin{aligned}
A_z &= -\int_0^x F_y(x',y,z)\,dx'+C_1(y,z)\\
&= -\int_0^x F_y(x',y,z)\,dx'+\int_0^y F_x(0,y',z)\,dy'.
\end{aligned}$$

(b) By direct differentiation, check that the $\mathbf{A}$ you obtained in part (a) satisfies $\nabla \times \mathbf{A} = \mathbf{F}$. Is $\mathbf{A}$ divergenceless? [This was a very asymmetrical construction, and it would be surprising if it were—although we know that there exists a vector whose curl is $\mathbf{F}$ and whose divergence is zero.]

Solution:

$$
\begin{aligned}
& \nabla \times \mathbf{A} \\
= & \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & \int_0^x F_z(x', y, z)\, dx' & -\int_0^x F_y(x', y, z)\, dx' + \int_0^y F_x(0, y', z)\, dy' \end{vmatrix} \\
= & \mathbf{i}\left(-\int_0^x \frac{\partial}{\partial y} F_y(x', y, z)\, dx' + \frac{\partial}{\partial y}\int_0^y F_x(0, y', z)\, dy' - \int_0^x \frac{\partial}{\partial z} F_z(x', y, z)\, dx'\right) \\
& -\mathbf{j}\frac{\partial}{\partial x}\left(-\int_0^x F_y(x', y, z)\, dx' + \int_0^y F_x(0, y', z)\, dy'\right) + \mathbf{k}\frac{\partial}{\partial x}\int_0^x F_z(x', y, z)\, dx' \\
= & \mathbf{i}\left(-\int_0^x \left(\frac{\partial}{\partial y} F_y(x', y, z) + \frac{\partial}{\partial z} F_z(x', y, z)\right) dx' + F_x(0, y, z)\right) \\
& +\mathbf{j}\frac{\partial}{\partial x}\left(\int_0^x F_y(x', y, z)\, dx'\right) + \mathbf{k}\frac{\partial}{\partial x}\int_0^x F_z(x', y, z)\, dx' \\
= & \mathbf{i}\left(\int_0^x \frac{\partial}{\partial x} F_x(x', y, z)\, dx' + F_x(0, y, z)\right) + \mathbf{j}\frac{\partial}{\partial x}\left(\int_0^x F_y(x', y, z)\, dx'\right) + \mathbf{k}\frac{\partial}{\partial x}\int_0^x F_z(x', y, z)\, dx' \\
= & \mathbf{i}F_x(x, y, z) + \mathbf{j}F_y(x, y, z) + \mathbf{k}F_z(x, y, z) = \mathbf{F}
\end{aligned}
$$

$$
\begin{aligned}
& \nabla \cdot \mathbf{A} \\
= & \int_0^x \frac{\partial}{\partial y} F_z(x', y, z)\, dx' - \int_0^x \frac{\partial}{\partial z} F_y(x', y, z)\, dx' + \int_0^y \frac{\partial}{\partial z} F_x(0, y', z)\, dy' \\
\neq & 0,
\end{aligned}
$$

in general.

(c) As an example, let $\mathbf{F} = y\hat{\mathbf{x}} + z\hat{\mathbf{y}} + x\hat{\mathbf{z}}$. Calculate $\mathbf{A}$, and confirm that $\nabla \times \mathbf{A} = \mathbf{F}$. (For further discussion see Prob. 5.51.)

Solution: Let $\mathbf{F} = y\hat{\mathbf{x}} + z\hat{\mathbf{y}} + x\hat{\mathbf{z}}$,

$$A_y = \int_0^x F_z(x', y, z)\, dx' = \int_0^x x' dx' = \frac{1}{2}x^2,$$

$$A_z = -\int_0^x z dx' + \int_0^y y' dy' = -xz + \frac{1}{2}y^2.$$

$$
\begin{aligned}
\mathbf{A} &= \frac{1}{2}x^2\hat{\mathbf{y}} + \left(\frac{1}{2}y^2 - xz\right)\hat{\mathbf{z}}, \\
\nabla \times \mathbf{A} &= \left(\frac{\partial}{\partial y}A_z - \frac{\partial}{\partial z}A_y\right)\hat{\mathbf{x}} + \left(-\frac{\partial}{\partial x}A_z\right)\hat{\mathbf{y}} + \left(\frac{\partial}{\partial x}A_y\right)\hat{\mathbf{x}} \\
&= y\hat{\mathbf{x}} + z\hat{\mathbf{y}} + x\hat{\mathbf{z}}.
\end{aligned}
$$

Problem 4 (Griffiths 5.36)

Find the magnetic dipole moment of the spinning spherical shell in Ex. 5.11. Show that for points $r > R$ the potential is that of a perfect dipole.

Solution:

$$
\begin{aligned}
\mathbf{m} &= \int d\mathbf{m} = \int I d\mathbf{A} = \int \frac{dq}{dt} d\mathbf{A} = \hat{\mathbf{z}}\int_0^\pi \frac{\sigma(2\pi R\sin\theta) R d\theta}{\frac{2\pi}{\omega}} \cdot \pi(R\sin\theta)^2 \\
&= \hat{\mathbf{z}}\sigma R^4\omega\pi\int_0^\pi \sin^3\theta d\theta = \frac{4\pi}{3}\sigma R^4\omega\hat{\mathbf{z}}.
\end{aligned}
$$

For points $r > R$ the potential is

$$\mathbf{A}\left(r,\theta,\varphi\right)|_{r>R} = \frac{\mu_0 R^4 \omega \sigma}{3}\frac{1}{r^2}\sin\theta\hat{\phi}.$$

$$\mathbf{A}_{dip} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi}\frac{4\pi}{3}\frac{\sigma R^4 \omega}{r^2}\hat{\mathbf{z}}\times\hat{\mathbf{r}} = \frac{\mu_0 R^2 \omega \sigma}{3}\frac{1}{r^2}\sin\theta\hat{\phi} = \mathbf{A}\left(r,\theta,\varphi\right)|_{r>R}.$$