Lect 1: Moving charges and magnetic fields (I)

Purcell, Chap 5

§1 History of magnetism

慈石召铁

magnetic stone attracts iron.

慈 → 磁

around 4~5 BC, ancient Chinese and Greeks discovered that lodestones attract iron. Later Chinese invented Compass for navigation, which was (1BC ~ 1AD) the first application of magnetism.

1600 W. Gilbert De Magnete: summary of ancient knowledge.

☆ 1819-1820 H. C. Oersted: electric currents results in magnetic fields.

revolutionary discovery: electricity ⟷ magnetism

Ampere's law:

☆ 1831 Faraday: electromagnetic induction:

Faraday's dairy

3000 "No!" → "Yes"

A changing magnetic field induces an electric field!

☆ Maxwell: A changing electric field induces an magnetic field! ⟷ Displacement current

also Ampere.

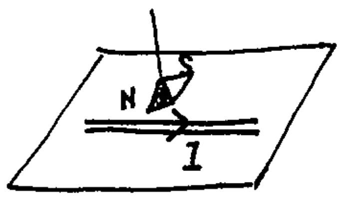
Complete Maxwell's equations.

Hertz: experimentally test the propogation of EM waves

☆ 1905 Einstein: Generalize the Lorentz invariance of Maxwell's equations to mechanical laws ⟷ Special Relativity!

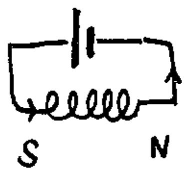
R.P. Feynman: The discovery of Maxwell's equations is more important than American's civil war in the long time scale, which took place approximately at the same time.

§2 magnetic forces



Oersted

Ampere replaced magnetic needle with a coil with current.

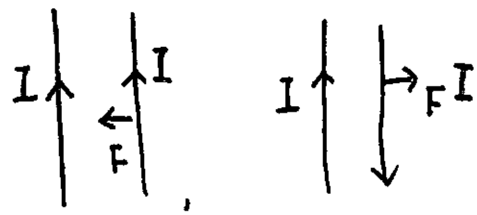


⟷ ∴ magnetic needle has "molecular current".

$\boxed{\text{Mach's puzzle:}}$

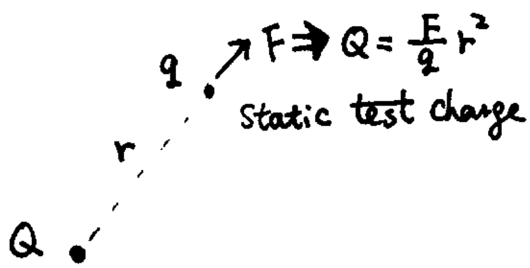
when the needle and wire are in the same plane, it seems that the system has the reflection symmetry. Why the needle rotates in a unique way?

$$\vec{F} = q\vec{E} + \underbrace{\frac{q}{c}\vec{v}\times\vec{B}}_{\text{Lorentz force}}$$

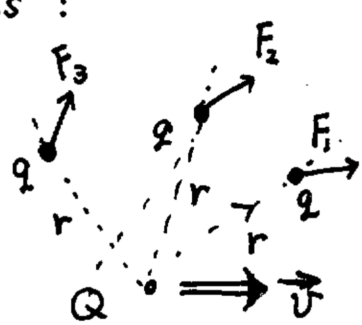


This equation can be viewed as the definition of the $\vec{B}$ field. But why can it be done like this? We will answer it step by step.

Question 1: how to measure moving charges?



Static charge

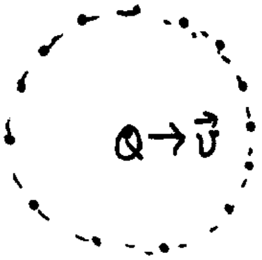


moving charge

$\boxed{\text{static test charge}}$

$$Q = \frac{?}{q} r^2$$

there's no reason for F along the radial direction!

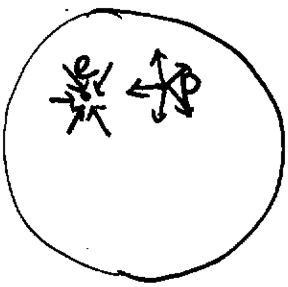


Let's use a shell of static test charge, at the moment that Q passes the center, measure the average force on the test charges

$$\frac{1}{4\pi}\oint \vec{E}\cdot d\vec{S} = Q ,$$

Gauss's law is valid even for Q is moving! An experiment fact.

Electric flux is independent of the motion of charges !!



before and after $\underline{e}$ and $\underline{p}$ form a H atom, does the total flux change?

Inside the atom, "e" moves much faster than "p".

suppose we have 2 electrons and 2 protons → one H_2 molecule
charge neutral

He atom
2 electrons
2 protons
2 neutrons
also charge neutral.

Charge is different from mass !

Q is also independent of reference frames ⇒ relativity scalar.

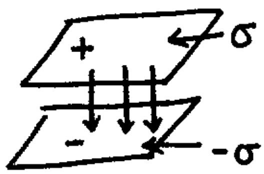
Charge conservation:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\vec{j} = 0 \quad , \quad \text{or} \quad \frac{\partial Q}{\partial t} + \iint \vec{j}\cdot d\vec{S} = 0$$

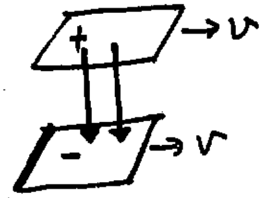
Charge invariance:

Charge is a relativistic scalar, which does depend on its motion and reference frame.

§3. Electric fields in different frames.



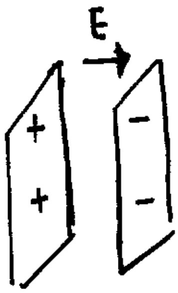
$E_z = 4\pi\sigma$,



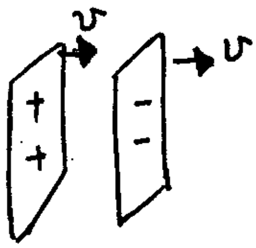
Ex: prove that even for the moving capacitor, "E" fields is still vertical.

$$E_z' = 4\pi\sigma' = 4\pi \frac{\sigma}{\sqrt{1-\beta^2}} = \frac{E_z}{\sqrt{1-\beta^2}}$$

$$Q = \sigma \cdot L^2 = \sigma' L \cdot L\sqrt{1-\beta^2} \Rightarrow \sigma' = \frac{\sigma}{\sqrt{1-\beta^2}}$$



$E_x = 4\pi\sigma$



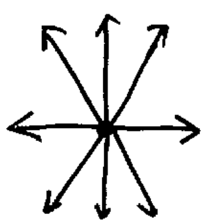
$E_x' = 4\pi\sigma' = E_x$

$\sigma' = \sigma$

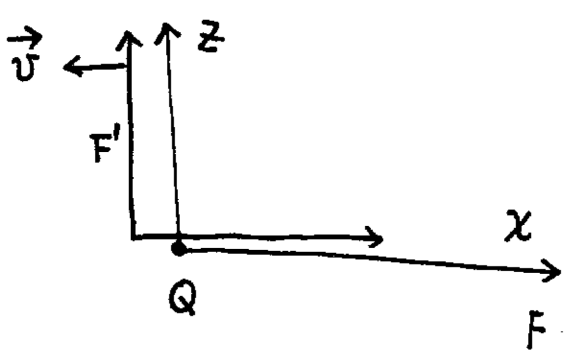
Conclusion: the reference frame F, where $\vec{E}$ is generated from the static charges, then suppose that frame F', is moving with the velocity $\vec{v}$ respect to F. We decompose $E_{\parallel}$, $E_{\parallel}'$ and $E_{\perp}$, $E_{\perp}'$ as parallel and perpendicular components to $\vec{v}$.

$$\boxed{E_{\parallel}' = E_{\parallel} \qquad \text{and} \qquad E_{\perp}' = \gamma E_{\perp}\ , \text{ where } \gamma = \frac{1}{\sqrt{1-\beta^2}}}$$

Electric fields from a moving charge : (XZ-plane)



static



Q is static at the origin of the F-frame

$$E_x = \frac{Q x}{(\sqrt{x^2+z^2})^3}$$

$$E_z = \frac{Q z}{(\sqrt{x^2+z^2})^3}$$

Suppose Frame F' is moving along $-\hat{x}$ at the speed of v. $\Rightarrow$

$$\begin{pmatrix} x \\ \cancel{c}t \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix},$$

then Q is moving at v along $\hat{x}$ in the frame of F'.

At Q passes the origin of F', $t'=0$. At this moment of t'

$\Rightarrow \begin{cases} x = \gamma x' \\ z = z' \end{cases}$, the Fields measured in F' should be

$$E'_x(\vec{r}', t'=0) = E_x(\vec{r}, t) = \frac{Qx}{(x^2+z^2)^{3/2}} = \frac{\gamma Q x'}{[(\gamma x')^2 + z'^2]^{3/2}}$$

$$E'_z(\vec{r}', t'=0) = \gamma E_z(\vec{r}, t) = \frac{\gamma Q z}{(x^2+z^2)^{3/2}} = \frac{\gamma Q z'}{[(\gamma x')^2 + z'^2]^{3/2}}$$

$$\Rightarrow \frac{E'_z(\vec{r}', t'=0)}{E'_x(\vec{r}', t'=0)} = \frac{z'}{x'} \Rightarrow$$

$\boxed{E' \text{ is along the radial direction in } F'}$

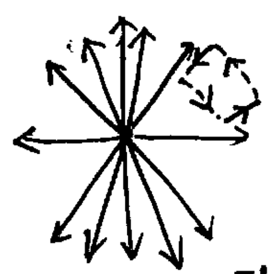
The total field

$$E'^2 = E'^2_x + E'^2_z \Rightarrow E'^2 = \frac{\gamma^2 Q^2 (x'^2 + z'^2)}{[(\gamma x')^2 + z'^2]^3} = \frac{Q^2 (x'^2 + z'^2)}{\gamma^4 [x'^2 + z'^2 (1-\beta^2)]^3}$$

$$= \frac{Q^2 \quad (1-\beta^2)^2}{(x'^2+z'^2)^2 \left(1 - \frac{\beta^2 z'^2}{x'^2+z'^2}\right)^3}$$

$$\Rightarrow \boxed{E' = \frac{Q}{r'^2} \frac{1-\beta^2}{(1-\beta^2 \sin^2\theta')^{3/2}}}$$

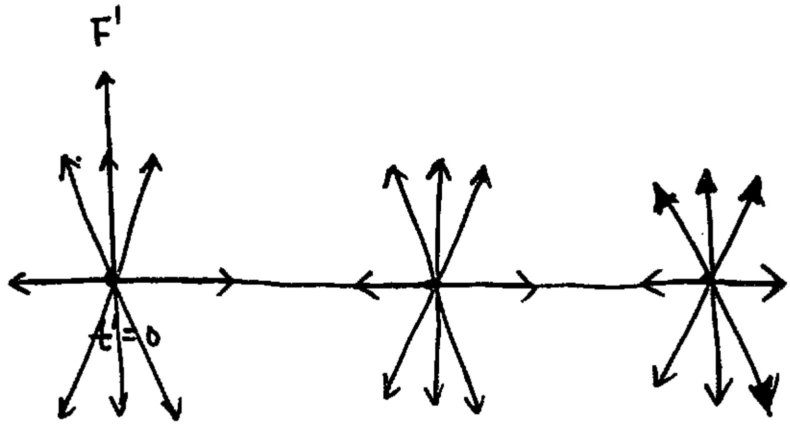
$E'_z = \frac{Q}{r'^2}\gamma$



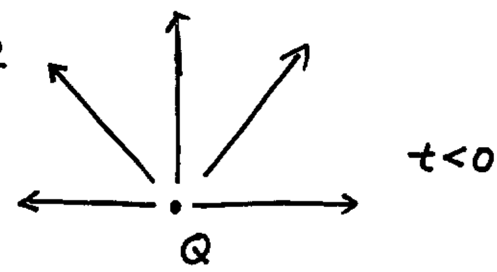
$\boxed{\oint \vec{E}' \cdot d\vec{l} \neq 0}$

$E'_x = \frac{Q}{r'^2}\gamma^{-2}$

E' isn't static field.

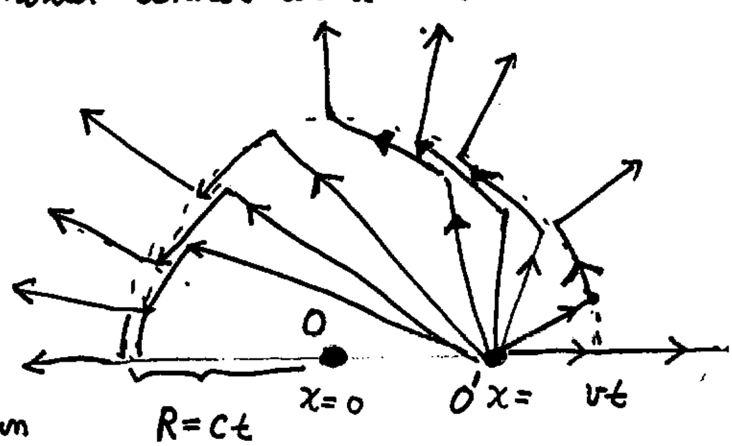


§ Electric fields of a sudden moving charge



at $t < 0$, charge Q is at rest at the origin.

at $\Delta t > t > 0$, charge Q accelerates in a short time interval Δt, and then it velocity becomes v along $\hat{x}$ direction, at $t > \Delta t$. Δt is a very short. Then at the distance $R > ct$, the eletric fields should not notice the motion of Q, thus should be the same as those at $t < 0$. At the distance $R < ct$, the fieldlines should be those of a moving charge at velocity v. The two different types of field lines should connet at a thin shell with the thickness $c\Delta t$.

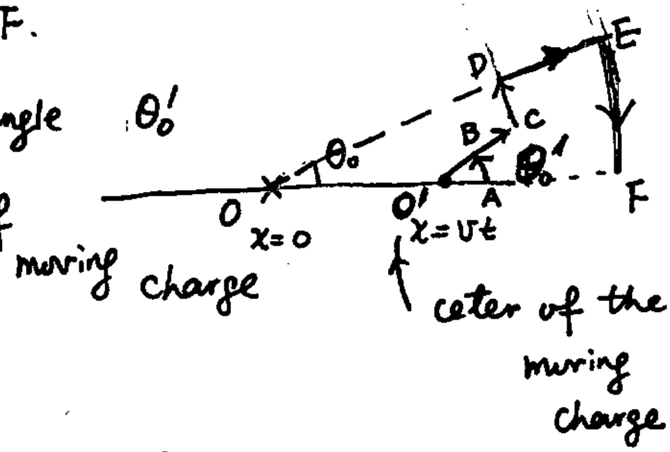


Thus in the thin shell with the radius of $R = ct$, and the thickness of $c\Delta t$, the E fields are along the polar direction from the right pole to left pole.

These fields are transverse field, which are different from the electrostatic fields which are longitudinal.

We need to decide how the two different regions are connected.

consider the region of $A \to B \to C \to D \to E \to F$.



AB is the area which span the polar angle θ_0'

respect to O'. the fields on AB are of

$$\int_0^{\theta_0'} \sin\theta' d\theta' \cdot 2\pi Q \frac{1-\beta^2}{(1-\beta^2\sin^2\theta')^{3/2}}$$

The flux passes the area span by E•F $\Rightarrow$ $\int_0^{\theta_0} \sin\theta \, d\theta \cdot 2\pi Q$

all other area don't antribute flux $\Rightarrow$

$$\int_0^{\theta_0'} \frac{\sin\theta' d\theta'}{(1-\beta^2\sin^2\theta')^{3/2}} (1-\beta^2) = \int_0^{\theta_0} \sin\theta \, d\theta$$

$$\int \frac{\sin\theta' d\theta'}{(1-\beta^2\sin^2\theta')^{3/2}} = -\int \frac{d\cos\theta'}{(1-\beta^2+\beta^2\cos^2\theta')^{3/2}} = -\beta^{-3}\int \frac{d\cos\theta'}{(\cos^2\theta' + \frac{1-\beta^2}{\beta^2})^{3/2}}$$

$$\int \frac{dx}{(a^2+x^2)^{3/2}} = \frac{x}{a^2(a^2+x^2)^{1/2}} + C$$

$$\Rightarrow \int_0^{\theta_0'} \frac{\sin\theta' d\theta'}{(1-\beta^2\sin^2\theta')^{3/2}} = -\beta^{-3} \left. \frac{\cos\theta'}{\frac{1-\beta^2}{\beta^2}\left(\cos^2\theta' + \frac{1-\beta^2}{\beta^2}\right)^{1/2}} \right|_0^{\theta_0'}$$

$$= \beta^{-1}\left[\frac{\beta}{1-\beta^2} - \frac{\cos\theta_0'}{1-\beta^2\left(\cos^2\theta_0' + \frac{1-\beta^2}{\beta^2}\right)^{1/2}}\right]$$

$$\Rightarrow \beta^{-1}\left[\beta - \frac{\cos\theta_0'}{\beta^{-1}[1-\beta^2\sin^2\theta_0']^{1/2}}\right] = [1-\cos\theta_0] \Rightarrow \boxed{\cos\theta_0 = \frac{\cos\theta_0'}{\sqrt{1-\beta^2\sin^2\theta_0'}}}$$

or $\boxed{\tan\theta_0' = \gamma\tan\theta_0}$

If we consider field lines like a rod, and every rod represents the same amount of flux, then the rods associated with the moving charge are steeper than those connecting to the rest position of the charge. Their relation is

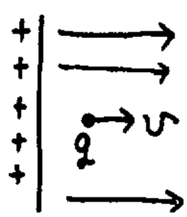
$$\boxed{\tan\theta_0' = \gamma \tan\theta_0}$$

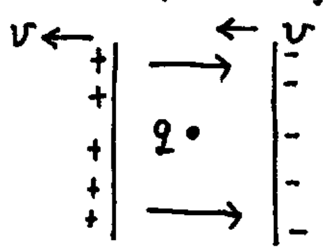
§ Forces on a moving charge without B field

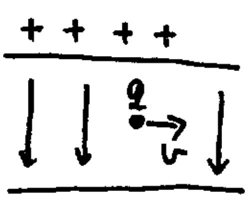
we have assumed that for a static charge $\vec{F} = q\vec{E}$. We can actually show that for a moving charge, its electric force remains $\vec{F}_e = q\vec{E}$, (it may contains additional Lorentz force part which depends on velocity $\vec{v}$).

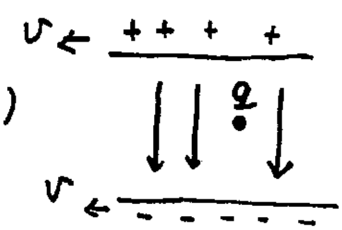
Let us consider two different frames: in the lab frame F, particle is moving and electric field is static. In the particle's frame F', particle is static, but electric field isn't.

F (lab frame) F' (particle frame)

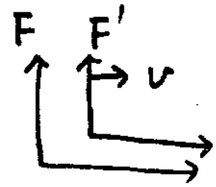
A)  E

A')  E'

B)  E

B')  E'

$$\begin{pmatrix} P_x \\ E/c \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} P_x' \\ E'/c \end{pmatrix}$$



$P_x > P_x'$

& $$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix}$$

$$c\,dt = \beta\gamma\,dx' + \gamma c\,dt' = \left[\beta\gamma \frac{dx'}{dt'} + \gamma c\right] dt'$$

$$\Rightarrow \quad dt = (1+\beta\beta')\gamma\,dt' \quad \text{where } \beta' = \frac{1}{c}\frac{dx'}{dt'} = \frac{v}{c}$$

$$dP_x = \gamma\,dP_x' + \gamma\beta/c\,dE' = \gamma\left(1 + \beta\frac{1}{c}\frac{dE'}{dP_x'}\right) dP_x' \xleftarrow{\text{check}} \frac{dE'}{dP_x'} = \frac{c^2 P_x'}{E'} = \beta'$$

$$\Rightarrow \quad dP_x = \gamma(1+\beta\beta')\,dP_x' \quad \Rightarrow \frac{dP_x}{dt} = \frac{dP_x'}{dt'} \quad \& \quad \frac{dP_y}{dt} = \frac{1}{\gamma}\frac{dP_y'}{dt'}$$

or: For two frames F & F', a particle is at rest in F', and F' moves at a velocity v respect to F. Decompose forces parallel and pendicular to $\vec{v}$ as $F_{\parallel}, F_{\parallel}'$ and $F_{\perp}, F_{\perp}'$, their relation

$$\boxed{\begin{aligned} \frac{dP_{\parallel}}{dt} &= \frac{dP_{\parallel}'}{dt'} \\ \frac{dP_{\perp}}{dt} &= \frac{1}{\gamma}\frac{dP_{\perp}'}{dt'} \end{aligned}}$$

, which is the same as E.

then
$$\frac{dP_{\parallel}'}{dt'} = qE_{\parallel}' = qE_{\parallel} = \frac{dP_{\parallel}}{dt}$$
$$\frac{dP_{\perp}'}{dt'} = qE_{\perp}' = \gamma q E_{\perp} = \gamma\frac{dP_{\perp}}{dt}$$
$$\Rightarrow \quad \frac{dP_{\parallel}}{dt} = qE_{\parallel}, \qquad \frac{dP_{\perp}}{dt} = qE_{\perp} \quad \checkmark$$

§ forces on a moving charge with B field

Let us consider a situation where $\vec{E}$ is zero everywhere, the test charge may still feel a velocity dependent force $\vec{F}_L(\vec{v})$. Generally speaking,

$F_{L,i} = T_{ij} v_j$, where T_{ij} should be rank-2 3-tensor. Moreover, we expect that our system is an conservative system $\vec{F}\cdot\vec{v} = 0 \Rightarrow T_{ij} = -T_{ji}$. only 3-independent component.

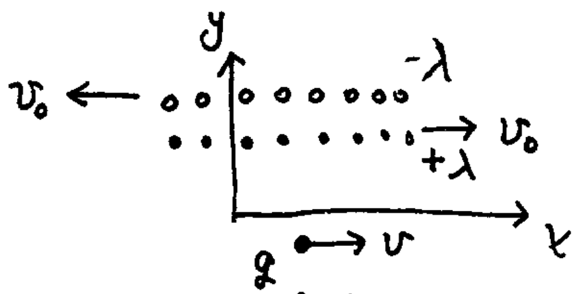
This can be represented as a 3-axial vector

$$B_i = \frac{1}{2}\epsilon_{ijk} T_{jk} \quad \Rightarrow \quad \vec{F}_L = \frac{q}{c}\vec{v}\times\vec{B}.$$

The form of Lorentz force should be viewed as an experiment fact, rather than being derived. Nevertheless, we will give an explanation based on Lorentz transform

Lab frame: A line of positive charge moving at the speed of v_0 to the right
negative ... left.

In this frame, the line charge densities are $\pm\lambda$, respectively. Thus total charge is zero, no electric fields.



put a test charge moving at the speed of v to the right. What is the force on q?

Let us change to the frame F' in which the test charge is at rest. Then the line charge densities $\lambda_\pm$ are not equal any more due to different contraction. In this frame F', the velocities of positive/negative charges are different

$$v_+' = \frac{v_0 - v}{1 - \frac{v_0 v}{c^2}} \qquad v_-' = \frac{v_0 + v}{1 + \frac{v_0 v}{c^2}} \qquad \text{define } \beta_\pm' = \frac{v_\pm'}{c} \quad \beta_0 = \frac{v_0}{c}$$

$$\Rightarrow \boxed{\beta_\pm' = \frac{\beta_0 \mp \beta}{1 \mp \beta_0\beta}}$$

For the positive charge, its line charge density in its rest frame should be $\lambda_{0,+} = \frac{\lambda}{\gamma_0}$; similarly $\lambda_{0,-} = \frac{-\lambda}{\gamma_0}$

$\Rightarrow$ in the frame F' $\Rightarrow$ $\lambda'_{\pm} = \lambda_{0,\pm}\, \gamma'_{\pm} = \pm \frac{\lambda}{\gamma_0} \gamma'_{\pm}$

the net charge density $\Delta\lambda' = \lambda'_+ - \lambda'_- = \frac{\lambda}{\gamma_0}\left[\gamma'_+ - \gamma'_-\right]$

$$\gamma'_+ - \gamma'_- = \frac{1}{\sqrt{1-\left(\frac{\beta_0-\beta}{1-\beta\beta_0}\right)^2}} - \frac{1}{\sqrt{1-\left(\frac{\beta_0+\beta}{1-\beta_0\beta}\right)^2}} = \frac{-2\beta_0\beta}{\sqrt{(1-\beta_0^2)(1-\beta^2)}} = -2\beta_0\beta\gamma_0\gamma$$

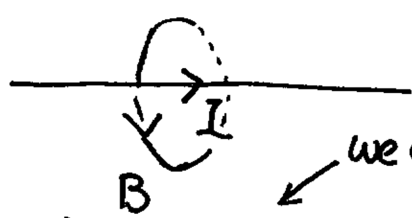
$$\Rightarrow \quad \Delta\lambda' = -2\lambda\beta_0\beta\gamma \Rightarrow E_y' = \frac{2\Delta\lambda}{r} = -\frac{4\lambda\gamma v v_0}{rc^2}$$

$$F_y' = \frac{4q\gamma\lambda v v_0}{rc^2}$$

in the Frame F $\Rightarrow$ $F_y = \frac{1}{\gamma} F_y' = \frac{4q\lambda v v_0}{rc^2} = \boxed{\frac{2I}{rc}}\ \frac{qv}{c}$

where $I = 2\lambda v_0$

$$\boxed{\text{Ampere's law} \quad B = \frac{2I}{rc}}$$



we don't have magnetic-monopole.

Magnetic field from electric charge is indeed an relativitic effect. Because electric force usually are cancelled due to charge neutrality, magnetic force can appear! E & M are naturally relativistic, although peopele didn't realize it until Einstein pointed it out!