Problem 3.4

Suppose there were two solutions to Poisson equation

$$\nabla^2 V_1 = \frac{\rho}{\epsilon_0}, \quad \nabla^2 V_2 = \frac{\rho}{\epsilon_0}$$

Let $V_3 = V_1 - V_2$. Then

$$\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = \frac{\rho}{\epsilon_0} - \frac{\rho}{\epsilon_0} = 0$$

That is, V_3 satisfies Laplacian equation.

Let $\vec{E}_3 = -\nabla V_3$ and calculate $\nabla \cdot (V_3 \vec{E}_3)$

$$\nabla \cdot (V_3 \vec{E}_3) = V_3 \nabla \cdot \vec{E}_3 + \vec{E}_3 \cdot (\nabla V_3)$$

$$= V_3 (-\nabla^2 V_3) + \vec{E}_3 \cdot (-\vec{E}_3)$$

$$= V_3 \times 0 - E_3^2 = -E_3^2$$

$$\Rightarrow \quad \nabla \cdot (V_3 \vec{E}_3) = -E_3^2 \quad \text{——— ①}$$

Integrate ① over the volume inside the boundary

$$\int_V \nabla \cdot (V_3 \vec{E}_3)\, d\tau = \int_V -E_3^2\, d\tau$$

$$\Rightarrow \int_S V_3 \vec{E}_3 \cdot d\vec{a} = -\int_V E_3^2\, d\tau \quad \text{——— ②}$$

The left hand side of ② is a surface integral over the boundary. And on the boundary,

$$E_3 = \frac{-\partial V_3}{\partial n} = \frac{\partial V_1}{\partial n} - \frac{\partial V_2}{\partial n} \quad (\text{on } S)$$

We know $\frac{\partial V}{\partial n}$ is specified on S.

$$\Rightarrow \frac{\partial V_1}{\partial n} = \frac{\partial V_2}{\partial n}$$

$$\Rightarrow E_3 = \frac{\partial V_1}{\partial n} - \frac{\partial V_2}{\partial n} = 0 \quad (\text{on } S)$$

$$\Rightarrow \int_S V_3 \vec{E_3} \cdot d\vec{a} = 0 \quad —— ③$$

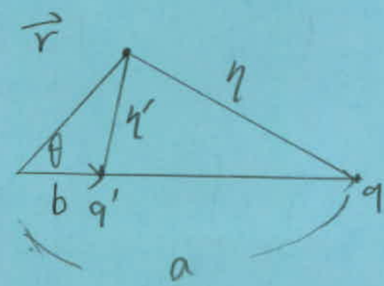
Putting ③ in ②

$$\Rightarrow \quad 0 = -\int_V E_3^2 \, d\tau \quad —— ④$$

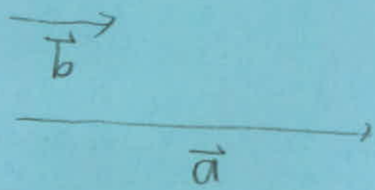
Since E_3^2 is never negative, ④ is true only if $E_3 = 0$, which means $\vec{E_3} = \vec{E_1} - \vec{E_2} = 0 \Rightarrow \vec{E_1} = \vec{E_2}$.

The field is uniquely determined.

Problem 3.7.



$$\vec{\eta}' = \vec{r} - \vec{b}$$



$$\begin{cases} q' = -\dfrac{R}{a} q \\ b = \dfrac{R^2}{a} \end{cases}$$

$$\Rightarrow \eta'^2 = r^2 + b^2 - 2\vec{r}\cdot\vec{b}$$
$$= r^2 + b^2 - 2rb\cos\theta$$

$$\Rightarrow \frac{1}{\eta'} = \frac{1}{\sqrt{r^2+b^2-2rb\cos\theta}} \quad \text{—— ①}$$

$$\vec{\eta} = \vec{r} - \vec{a}$$

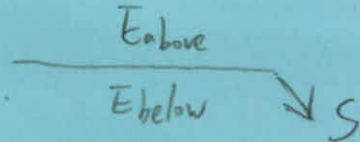
$$\Rightarrow \eta^2 = r^2 + a^2 - 2ra\cos\theta$$

$$\Rightarrow \frac{1}{\eta} = \frac{1}{\sqrt{r^2+a^2-2ra\cos\theta}} \quad \text{—— ②}$$

(a) $$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\eta} + \frac{q'}{\eta'}\right) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2+a^2-2aR\cos\theta}} + \frac{-\frac{R}{a}q}{\sqrt{r^2+b^2-2bR\cos\theta}}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{r^2+a^2-2aR\cos\theta}} - \frac{q}{\sqrt{R^2+\left(\frac{ra}{R}\right)^2-2ra\cos\theta}}\right]$$

(b) $$E_{above} - E_{below} = \frac{\sigma}{\epsilon_0}$$



$$E_{above} = -\frac{\partial V}{\partial r}\Big|_{r=R}$$

$E_{below} = 0$ (inside metal)

$$\Rightarrow \sigma = -\epsilon_0 \frac{\partial V}{\partial r}\Big|_{r=R}$$

$$= \frac{-q}{4\pi}\left[\frac{-\frac{1}{2}(2r-2a\cos\theta)}{(r^2+a^2-2ra\cos\theta)^{\frac{3}{2}}} - \frac{-\frac{1}{2}(2\frac{a^2r}{R^2}-2a\cos\theta)}{(R^2+(\frac{ra}{R})^2-2ra\cos\theta)^{\frac{3}{2}}}\right]\Big|_{r=R}$$

$$= \frac{-q}{4\pi} \times \frac{(-R+\frac{a^2}{R})}{(R^2+a^2-2Ra\cos\theta)^{\frac{3}{2}}} = \frac{q}{4\pi}\frac{(R-\frac{a^2}{R})}{(R^2+a^2-2aR\cos\theta)^{\frac{3}{2}}}$$

$$Q = \int \sigma\, da = \int_0^{2\pi}\int_0^{\pi} \sigma\, R^2 \sin\theta\, d\theta\, d\phi$$

$$= 2\pi R^2 \int_0^{\pi} \sigma \sin\theta\, d\theta$$

$$= \frac{qR^2}{2}(R-\frac{a^2}{R})\int_0^{\pi}(R^2+a^2-2aR\cos\theta)^{\frac{-3}{2}}\sin\theta\, d\theta \quad \text{Let } x=\cos\theta$$

$$= \frac{qR^2}{2}(R-\frac{a^2}{R})\int_{-1}^{1}(R^2+a^2-2aRx)^{-\frac{3}{2}}dx$$

$$= \frac{qR^2}{2}(R-\frac{a^2}{R})(\frac{1}{aR})(R^2+a^2-2aRx)^{-\frac{1}{2}}\Big|_{-1}^{1}$$

$$= \frac{qR}{2a}(R-\frac{a^2}{R})\left[\frac{1}{a-R} - \frac{1}{R+a}\right] = \frac{-qR}{a} = q'$$

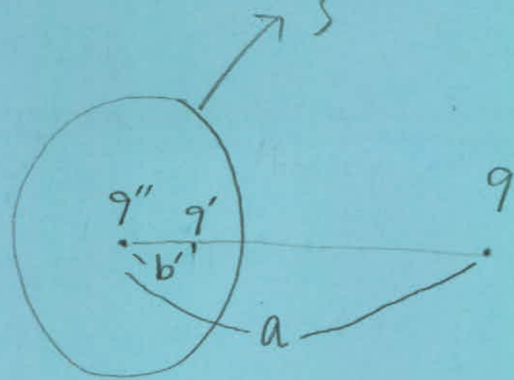
(c) Let's use eq. 2-43. $W = \frac{1}{2}\int \rho V d\tau$

$$W = \frac{1}{2}\left[\int_S \sigma V da + qV\right] \quad (V \text{ on } S \text{ is zero})$$

$$= \frac{1}{2}qV = \frac{1}{2}q\cdot\left(\frac{1}{4\pi\epsilon_0}\frac{q'}{\eta'}\right)\Big|_{r=a,\theta=0} = -\frac{1}{8\pi\epsilon_0}\frac{q^2R}{(a^2-R^2)}$$

Problem 3.8

Let q'' be the second image charge.



The potential on S contributed by q'' must be uniform, so q'' can only be at the center. And

$$V_0 = \frac{1}{4\pi\epsilon_0}\frac{q''}{R} \Rightarrow q'' = 4\pi\epsilon_0 R V_0$$

The field on q are from q' and q''

$$\vec{E}_q = \frac{1}{4\pi\epsilon_0}\left(\frac{q''}{a^2}\hat{r} + \frac{q'}{(a-b)^2}\hat{r}\right), \text{ with } q' = -\frac{R}{a}q$$

$$b = \frac{R^2}{a}$$

For a neutral sphere $q'' = -q'$

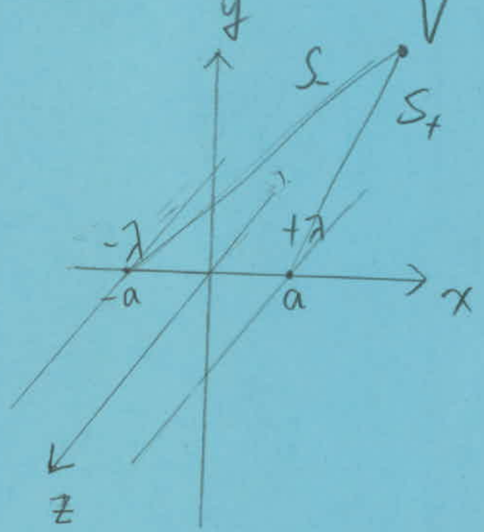
$$\vec{F}_q = q\vec{E}_q = \frac{q}{4\pi\epsilon_0} \times \frac{\frac{R}{a}q}{a^2}\hat{r} + \frac{1}{4\pi\epsilon_0}\frac{-\frac{R}{a}q^2}{\left(a-\frac{R^2}{a}\right)^2}\hat{r}$$

$$= -\frac{q^2}{4\pi\epsilon_0}\left(\frac{R^3}{a^3}\right)\frac{(2a^2-R^2)}{(a^2-R^2)^2}\hat{r}$$

Problem 3-11.

Consider two line charges parallel to z. One is at $x=-a$ with $-\lambda$ and the other is at $x=a$ with λ.

Set $V=0$ at z-axis.



$$V = V_+ + V_-$$
$$= \frac{\lambda}{2\pi\epsilon_0}\left[-\ln\frac{S_+}{a} + \ln\frac{S_-}{a}\right],$$

$$S_+ = \sqrt{(x-a)^2+y^2} \quad \text{and} \quad S_- = \sqrt{(x+a)^2+y^2}$$

$$\Rightarrow V = \frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{(x+a)^2+y^2}{(x-a)^2+y^2}\right] \quad \text{—} \ (1)$$

We want to find the equipotential surfaces where potential is V_0

$$V_0 = \frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{(x+a)^2+y^2}{(x-a)^2+y^2}\right]$$

$$\Rightarrow \frac{(x+a)^2+y^2}{(x-a)^2+y^2} = e^{\frac{4\pi\epsilon_0 V_0}{\lambda}}. \quad \text{Let } k = e^{\frac{4\pi\epsilon_0 V}{\lambda}}$$

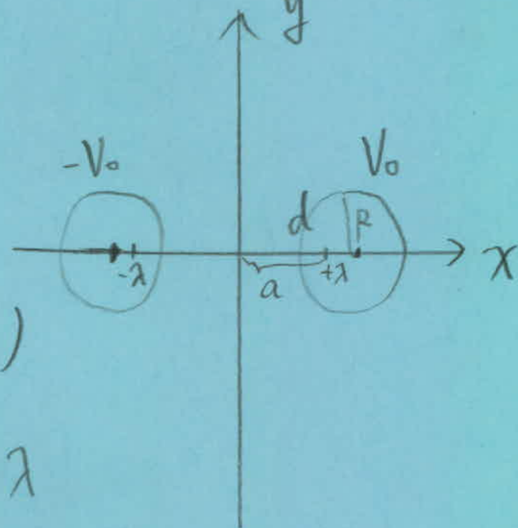
$$\Rightarrow (x+a)^2+y^2 = k[(x-a)^2+y^2] \Rightarrow \text{an equation of a circle.}$$

$\Rightarrow$ Do some algebra and you will obtain

$$(x-d)^2+y^2 = R^2, \quad \text{with} \begin{cases} d = a\left(\frac{k+1}{k-1}\right) \\ R = \frac{2a\sqrt{k}}{|k-1|} \end{cases}$$

In terms of V_0

$$\begin{cases} d = a\coth\left(\frac{2\pi\epsilon_0 V_0}{\lambda}\right) \\ R = a\,\mathrm{csch}\left(\frac{2\pi\epsilon_0 V_0}{\lambda}\right) \end{cases} \quad \text{——— ②}$$



We can replace the surface charges by these two line charges (image charges)

$x = a$ is the position of λ

$x = -a$ is the position of $-\lambda$.

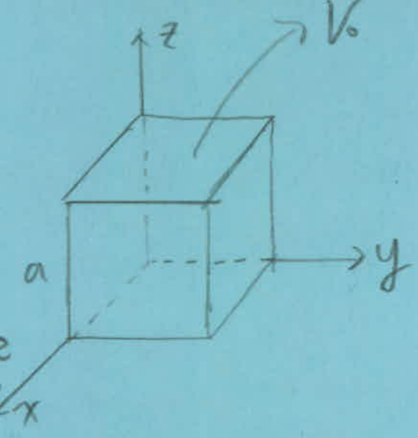
Solve ② for a and λ,

$$\begin{cases} a = \sqrt{d^2 - R^2} \\ \lambda = \dfrac{2\pi\epsilon_0 V_0}{\cosh^{-1}(d/R)} \end{cases} \quad \text{——— ③}$$

Putting ③ in ①, we obtain

$$V = \frac{V_0}{2\cosh^{-1}(d/R)} \ln\left[\frac{\left(x+\sqrt{d^2-R^2}\right)^2 + y^2}{\left(x-\sqrt{d^2-R^2}\right)^2 + y^2}\right]$$

Problem 3-15



This is the typical boundary problem. Inside the box, there is no charge, so the potential $V(x,y,z)$ satisfies the Laplacian equation,

$$\nabla^2 V = 0.$$

Laplacian equation together with the complete boundary conditions (potential on the six faces) has the uniqe solution.
For a cubic box, it is proper to express $\nabla^2 V = 0$ in the Cartesian coordinate,

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{——} \quad ①$$

Here, we use the method of separation of variables. (cf. 3-3)
Let the solution in the form:

$$V(x,y,z) = X(x)\,Y(y)\,Z(z) \quad \text{——} \quad ②$$

Putting ② in ①, we obtain

$$YZ\frac{\partial^2 X}{\partial x^2} + XZ\frac{\partial^2 Y}{\partial y^2} + XY\frac{\partial^2 Z}{\partial z^2} = 0. \quad \text{——} \quad ③$$

Dividing ③ by XYZ, we obtain

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial Z}{\partial z^2} = 0 \quad \text{—— ④}$$

Equation ④ is true only if the three terms are constants independent of x, y, or z.

Let these constants be $-k_x^2$, $-k_y^2$, $k_x^2+k_y^2$. Then

$$\begin{cases} \frac{1}{X^2}\frac{\partial^2 X}{\partial x^2} = -k_x^2 \\ \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k_y^2 \\ \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = \sqrt{k_x^2+k_y^2} \end{cases} \Rightarrow \begin{cases} \frac{\partial^2 X}{\partial x^2} = -k_x^2 X \\ \frac{\partial^2 Y}{\partial y^2} = -k_y^2 Y \\ \frac{\partial^2 Z}{\partial z^2} = k_x^2 + k_y^2 Z \end{cases} \quad \text{—— ⑤}$$

You can choose the three constants in the other forms if their sum is zero. We choose these forms: $-k_x^2$, $-k_y^2$, $k_x^2+k_y^2$, such that the solution looks simpler. As you can see in ⑤, $-k_x^2$ and $-k_y^2$ make the solution of X and Y be sinusoidal ($\sin k_x x$, $\cos k_x x$ ....). The sinusoidal functions better descibe the boundary conditions: $\begin{cases} V(x=0) = V(x=a) = 0 \\ V(y=0) = V(y=a) = 0. \end{cases}$

We call the type of boundary conditions: periodical boundary conditions.
On the other hand, $k_x^2+k_y^2$ make the solution of Z be $\sinh(\sqrt{k_x^2+k_y^2}\, x)$ and $\cosh(\sqrt{k_x^2+k_y^2}\, y)$, which better describe the boundary conditions $\begin{cases} V(z=0) = 0, \\ V(z=a) = V_0 \end{cases}$

Since there is a difference between $V(z=0)$ and $V(z=a)$.

The general solution of X is

$$X = A_1 \sin(k_x x) + A_2 \cos(k_x x)$$

Applying $V(x=0)=0$, we have

$$A_2 = 0$$

Applying $V(x=a)=0$, we have

$$\sin(k_x a) = 0$$

It follows that

$$k_x = \frac{m}{a}\pi, \quad (m = 1, 2, 3, \ldots)$$

$$\Rightarrow X = A_1 \sin\left(\frac{m}{a}\pi\right)$$

Y has the same boundary conditions, so

$$k_y = \frac{n\pi}{a}, \quad (n = 1, 2, 3 \ldots)$$

$$Y = B_1 \sin\left(\frac{n\pi}{a}\right)$$

For Z, its general solution is

$$Z = C_1 \sinh\left(\sqrt{k_x^2 + k_y^2}\, z\right) + C_2 \cosh\left(\sqrt{k_x^2 + k_y^2}\, z\right)$$

$$= C_1 \sinh\left(\sqrt{m^2+n^2}\, \frac{\pi z}{a}\right) + C_2 \cosh\left(\sqrt{m^2+n^2}\, \frac{\pi z}{a}\right)$$

Applying $V(z=0)=0$, we obtain $(\cosh(0)=1, \sinh(0)=0)$

$$C_2 = 0$$

Now, we have

$$Z = C_1 \sinh\left(\sqrt{m^2+n^2}\,\frac{\pi z}{a}\right).$$

and

$$V = XYZ = \underbrace{A_1 B_1 C_1}_{\text{Let it be } C} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right) \sinh\left(\sqrt{m^2+n^2}\,\frac{\pi z}{a}\right)$$

$$= C \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right) \sinh\left(\sqrt{m^2+n^2}\,\frac{\pi z}{a}\right) \quad \text{——} \ ⑥$$

We haven't used the condition: $V(x, y, z=a) = V_0$.

We know the number of m and n is infinite.

It suggests that we should include all m and n.

$$V = \sum_{m,n=1}^{\infty} C_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right) \sinh\left(\sqrt{m^2+n^2}\,\frac{\pi z}{a}\right) \quad \text{——} \ ⑦$$

Applying $V(x, y, z=a) = V_0$ in ⑦, we obtain

$$V_0 = \sum_{m,n=1}^{\infty} C_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right) \sinh\left(\sqrt{m^2+n^2}\,\pi\right) \quad \text{——} \ ⑧$$

To determine C_{mn}, we use the orthogonality of sin functions:

$$\int_0^a \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n'\pi x}{a}\right) dx = \frac{a}{2}\delta_{n,n'} \qquad \delta_{n,n'} = \begin{cases} 1, & n = n' \\ 0, & n \neq n' \end{cases}$$

Multiplying $\int_0^a \int_0^a \sin(\frac{m'\pi x}{a}) \sin(\frac{n'\pi y}{a})\, dx\, dy$ on the both sides of ⑧

and using the orthogonality on the right hand side, we obtain

$$V_0 \int_0^a \int_0^a \sin(\frac{m'\pi x}{a}) \sin(\frac{n'\pi y}{a})\, dx\, dy = \sum_{m,n=1}^{\infty} C_{mn} \frac{a\delta_{mm'}}{2} \cdot \frac{a\delta_{nn'}}{2} \cdot \sinh(\sqrt{m^2+n^2}\pi)$$

$$\Rightarrow V_0 \int_0^a \int_0^a \sin(\frac{m'\pi x}{a}) \sin(\frac{n'\pi y}{a})\, dx\, dy = \frac{a^2}{4} C_{m'n'} \sinh(\sqrt{m'^2+n'^2}\pi)$$

$$\Rightarrow C_{m'n'} = \frac{4V_0}{a^2 \sinh(\sqrt{m'^2+n'^2}\pi)} \int_0^a \int_0^a \sin(\frac{m'\pi x}{a}) \sin(\frac{n'\pi y}{a})\, dx\, dy$$

$$= \frac{4V_0}{a^2 \sinh(\sqrt{m'^2+n'^2}\pi)} \left(-\frac{a}{m'\pi} \cos\frac{m'\pi x}{a}\right)\Bigg|_0^a \cdot \left(-\frac{a}{n'\pi} \cos\frac{n'\pi y}{a}\right)\Bigg|_0^a$$

$$= \begin{cases} \dfrac{V_0}{\sinh(\sqrt{m'^2+n'^2}\pi)} \times \dfrac{16}{m'n'\pi^2}, & \text{both } m' \text{ and } n' \text{ are odd.} \\ 0, & \text{else} \end{cases}$$

Putting C_{mn} in ⑦, we get the final solution,

$$\boxed{V(x,y,z) = \sum_{m,n=1,3,5}^{\infty} \frac{16V_0}{m \cdot n\pi^2} \sin(\frac{m\pi}{a}x) \sin(\frac{n\pi y}{a}) \frac{\sinh(\sqrt{m^2+n^2}\frac{\pi z}{a})}{\sinh(\sqrt{m^2+n^2} \cdot \pi)}}$$

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