**Problem 1.39**

$$\begin{aligned}\nabla\cdot\mathbf{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\, r\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, r\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r\sin\theta\cos\phi)\\ &= \frac{1}{r^2}3r^2\cos\theta + \frac{1}{r\sin\theta}r\,2\sin\theta\cos\theta + \frac{1}{r\sin\theta}r\sin\theta(-\sin\phi)\\ &= 3\cos\theta + 2\cos\theta - \sin\phi = 5\cos\theta - \sin\phi\end{aligned}$$

$$\int(\nabla\cdot\mathbf{v})d\tau = \int(5\cos\theta - \sin\phi)\, r^2\sin\theta\, dr\, d\theta\, d\phi = \int_0^R r^2\,dr \int_0^{\frac{\pi}{2}}\left[\int_0^{2\pi}(5\cos\theta - \sin\phi)\,d\phi\right] d\theta\sin\theta$$

$$\hookrightarrow 2\pi(5\cos\theta)$$

$$= \left(\frac{R^3}{3}\right)(10\pi)\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\; d\theta$$

$$\hookrightarrow \left.\frac{\sin^2\theta}{2}\right|_0^{\frac{\pi}{2}} = \frac{1}{2}$$

$$= \boxed{\frac{5\pi}{3}R^3.}$$

Two surfaces—one the hemisphere: $d\mathbf{a} = R^2\sin\theta\, d\theta\, d\phi\,\hat{\mathbf{r}};\ r = R;\ \phi: 0\to 2\pi,\ \theta: 0\to\frac{\pi}{2}$.

$\int\mathbf{v}\cdot d\mathbf{a} = \int(r\cos\theta)R^2\sin\theta\, d\theta\, d\phi = R^3\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\, d\theta\int_0^{2\pi}d\phi = R^3\left(\frac{1}{2}\right)(2\pi) = \pi R^3$.

other the flat bottom: $d\mathbf{a} = (dr)(r\sin\theta\, d\phi)(+\hat{\boldsymbol{\theta}}) = r\, dr\, d\phi\,\hat{\boldsymbol{\theta}}$ (here $\theta = \frac{\pi}{2}$). $r: 0\to R,\ \phi: 0\to 2\pi$.

$\int\mathbf{v}\cdot d\mathbf{a} = \int(r\sin\theta)(r\,dr\,d\phi) = \int_0^R r^2\,dr\int_0^{2\pi}d\phi = 2\pi\frac{R^3}{3}$.

Total: $\int\mathbf{v}\cdot d\mathbf{a} = \pi R^3 + \frac{2}{3}\pi R^3 = \frac{5}{3}\pi R^3$. ✓

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**Problem 1.42**

(a)
$$\begin{aligned}\nabla\cdot\mathbf{v} &= \frac{1}{s}\frac{\partial}{\partial s}\left(s\, s(2+\sin^2\phi)\right) + \frac{1}{s}\frac{\partial}{\partial\phi}(s\sin\phi\cos\phi) + \frac{\partial}{\partial z}(3z)\\ &= \frac{1}{s}2s(2+\sin^2\phi) + \frac{1}{s}s(\cos^2\phi - \sin^2\phi) + 3\\ &= 4 + 2\sin^2\phi + \cos^2\phi - \sin^2\phi + 3\\ &= 4 + \sin^2\phi + \cos^2\phi + 3 = \boxed{8.}\end{aligned}$$

(b) $\int(\nabla\cdot\mathbf{v})d\tau = \int(8)s\,ds\,d\phi\,dz = 8\int_0^2 s\,ds\int_0^{\frac{\pi}{2}}d\phi\int_0^5 dz = 8(2)\left(\frac{\pi}{2}\right)(5) = \boxed{40\pi.}$

Meanwhile, the surface integral has five parts:

top: $z = 5,\ d\mathbf{a} = s\,ds\,d\phi\,\hat{\mathbf{z}};\ \mathbf{v}\cdot d\mathbf{a} = 3z\,s\,ds\,d\phi = 15s\,ds\,d\phi.\ \int\mathbf{v}\cdot d\mathbf{a} = 15\int_0^2 s\,ds\int_0^{\frac{\pi}{2}}d\phi = 15\pi.$

bottom: $z = 0,\ d\mathbf{a} = -s\,ds\,d\phi\,\hat{\mathbf{z}};\ \mathbf{v}\cdot d\mathbf{a} = -3z\,s\,ds\,d\phi\hat{\mathbf{z}} = 0.\ \int\mathbf{v}\cdot d\mathbf{a} = 0.$

back: $\phi = \frac{\pi}{2},\ d\mathbf{a} = ds\,dz\,\hat{\boldsymbol{\phi}};\ \mathbf{v}\cdot d\mathbf{a} = s\sin\phi\cos\phi\,ds\,dz = 0.\ \int\mathbf{v}\cdot d\mathbf{a} = 0.$

left: $\phi = 0,\ d\mathbf{a} = -ds\,dz\,\hat{\boldsymbol{\phi}};\ \mathbf{v}\cdot d\mathbf{a} = -s\sin\phi\cos\phi\,ds\,dz = 0.\ \int\mathbf{v}\cdot d\mathbf{a} = 0.$

front: $s = 2,\ d\mathbf{a} = s\,d\phi\,dz\,\hat{\mathbf{s}};\ \mathbf{v}\cdot d\mathbf{a} = s(2+\sin^2\phi)s\,d\phi\,dz = 4(2+\sin^2\phi)d\phi\,dz.$

$\int\mathbf{v}\cdot d\mathbf{a} = 4\int_0^{\frac{\pi}{2}}(2+\sin^2\phi)d\phi\int_0^5 dz = (4)(\pi + \frac{\pi}{4})(5) = 25\pi.$

So $\oint\mathbf{v}\cdot d\mathbf{a} = 15\pi + 25\pi = 40\pi$. ✓

(c)
$$\begin{aligned}\nabla\times\mathbf{v} &= \left(\frac{1}{s}\frac{\partial}{\partial\phi}(3z) - \frac{\partial}{\partial z}(s\sin\phi\cos\phi)\right)\hat{\mathbf{s}} + \left(\frac{\partial}{\partial z}\left(s(2+\sin^2\phi)\right) - \frac{\partial}{\partial s}(3z)\right)\hat{\boldsymbol{\phi}}\\ &\quad + \frac{1}{s}\left(\frac{\partial}{\partial s}(s^2\sin\phi\cos\phi) - \frac{\partial}{\partial\phi}\left(s(2+\sin^2\phi)\right)\right)\hat{\mathbf{z}}\\ &= \frac{1}{s}(2s\sin\phi\cos\phi - s\,2\sin\phi\cos\phi)\hat{\mathbf{z}} = \boxed{0.}\end{aligned}$$

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**Problem 1.48**

First method: use Eq. 1.99 to write $J = \int e^{-r}\left(4\pi\delta^3(\mathbf{r})\right)d\tau = 4\pi e^{-0} = \boxed{4\pi.}$

Second method: integrating by parts (use Eq. 1.59).

$$J = -\int_{\mathcal{V}}\frac{\hat{\mathbf{r}}}{r^2}\cdot\nabla(e^{-r})\,d\tau + \oint_{\mathcal{S}}e^{-r}\frac{\hat{\mathbf{r}}}{r^2}\cdot d\mathbf{a}.\quad\text{But}\quad \nabla\left(e^{-r}\right) = \left(\frac{\partial}{\partial r}e^{-r}\right)\hat{\mathbf{r}} = -e^{-r}\hat{\mathbf{r}}.$$

$$= \int\frac{1}{r^2}e^{-r}4\pi r^2\,dr + \int e^{-r}\frac{\hat{\mathbf{r}}}{r^2}\cdot r^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}} = 4\pi\int_0^R e^{-r}\,dr + e^{-R}\int\sin\theta\,d\theta\,d\phi$$

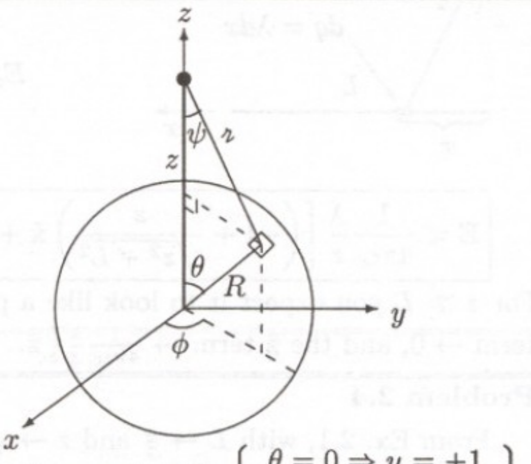
$$= 4\pi\left(-e^{-r}\right)\Big|_0^R + 4\pi e^{-R} = 4\pi\left(-e^{-R} + e^{-0}\right) = 4\pi.\checkmark\quad(\text{Here } R = \infty,\ \text{so } e^{-R} = 0.)$$

**Problem 2.7**

$\mathbf{E}$ is clearly in the $z$ direction. From the diagram,

$dq = \sigma da = \sigma R^2 \sin\theta \, d\theta \, d\phi,$
$\imath^2 = R^2 + z^2 - 2Rz\cos\theta,$
$\cos\psi = \frac{z - R\cos\theta}{\imath}.$

So




$$E_z = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma R^2 \sin\theta \, d\theta \, d\phi (z - R\cos\theta)}{(R^2 + z^2 - 2Rz\cos\theta)^{3/2}}. \qquad \int d\phi = 2\pi.$$

$$= \frac{1}{4\pi\epsilon_0}(2\pi R^2\sigma)\int_0^\pi \frac{(z - R\cos\theta)\sin\theta}{(R^2 + z^2 - 2Rz\cos\theta)^{3/2}}\,d\theta. \quad \text{Let } u = \cos\theta;\ du = -\sin\theta\,d\theta; \left\{\begin{array}{l}\theta = 0 \Rightarrow u = +1 \\ \theta = \pi \Rightarrow u = -1\end{array}\right\}.$$

$$= \frac{1}{4\pi\epsilon_0}(2\pi R^2\sigma)\int_{-1}^1 \frac{z - Ru}{(R^2 + z^2 - 2Rzu)^{3/2}}\,du. \quad \text{Integral can be done by partial fractions—or look it up.}$$

$$= \frac{1}{4\pi\epsilon_0}(2\pi R^2\sigma)\left[\frac{1}{z^2}\frac{zu - R}{\sqrt{R^2 + z^2 - 2Rzu}}\right]_{-1}^1 = \frac{1}{4\pi\epsilon_0}\frac{2\pi R^2\sigma}{z^2}\left\{\frac{(z - R)}{|z - R|} - \frac{(-z - R)}{|z + R|}\right\}.$$

For $z > R$ (outside the sphere), $E_z = \frac{1}{4\pi\epsilon_0}\frac{4\pi R^2\sigma}{z^2} = \frac{1}{4\pi\epsilon_0}\frac{q}{z^2}$, so $\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\,\hat{\mathbf{z}}.}$

For $z < R$ (inside), $E_z = 0$, so $\boxed{\mathbf{E} = 0.}$

**Problem 2.8**

According to Prob. 2.7, all shells *interior* to the point (i.e. at smaller $r$) contribute as though their charge were concentrated at the center, while all exterior shells contribute nothing. Therefore:

$$\mathbf{E}(r) = \frac{1}{4\pi\epsilon_0}\frac{Q_{\text{int}}}{r^2}\,\hat{\mathbf{r}},$$

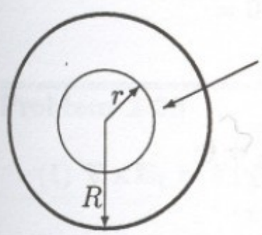
where $Q_{\text{int}}$ is the total charge interior to the point. *Outside* the sphere, *all* the charge is interior, so

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\,\hat{\mathbf{r}}.}$$

*Inside* the sphere, only that fraction of the total which is interior to the point counts:

$$Q_{\text{int}} = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3}Q = \frac{r^3}{R^3}Q, \quad \text{so} \quad \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{r^3}{R^3}Q\frac{1}{r^2}\,\hat{\mathbf{r}} = \boxed{\frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}\mathbf{r}.}$$

**Problem 2.12**




$$\oint \mathbf{E}\cdot d\mathbf{a} = E \cdot 4\pi r^2 = \frac{1}{\epsilon_0}Q_{\text{enc}} = \frac{1}{\epsilon_0}\frac{4}{3}\pi r^3\rho. \quad \text{So}$$

$$\boxed{\mathbf{E} = \frac{1}{3\epsilon_0}\rho r\,\hat{\mathbf{r}}.}$$

Since $Q_{\text{tot}} = \frac{4}{3}\pi R^3\rho$, $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}\mathbf{r}$ (as in Prob. 2.8).