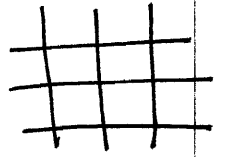


# Lect 5 Curvilinear Coordinates

$\vec{r} = x \hat{e}_x + y \hat{e}_y + z \hat{e}_z = \vec{r}(x, y, z)$  Cartesian coordinates

the grid  $x = \text{const}_1, y = \text{const}_2, z = \text{const}_3$  of orthogonal planes



unit vectors  $\hat{e}_x = \frac{\partial \vec{r}}{\partial x}, \hat{e}_y = \frac{\partial \vec{r}}{\partial y}, \hat{e}_z = \frac{\partial \vec{r}}{\partial z}$

are const unit vectors.

for many applications, we need curvi-linear coordinates.

e.g. the surface of the earth.

$\vec{r}(u^1, u^2, u^3)$ , the curves of  $u^1 = \text{const}_1$   
 surfaces  $u^2 = \text{const}_2$   
 $u^3 = \text{const}_3$

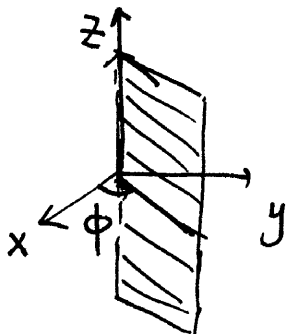
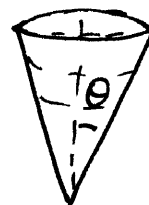
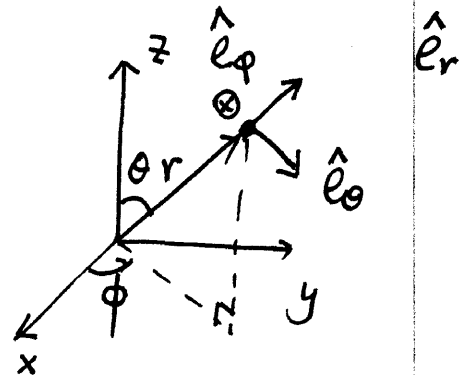
say, for the spherical coordinates

$\vec{r}(r, \theta, \varphi)$

$r = \text{const}_1$  specifies a spherical surface

$\theta = \text{const}_2$  specifies a cone surface

$\varphi = \text{const}_3$  specifies a half plane



unit vectors

$$\hat{e}_r = \frac{\partial \vec{r}}{\partial r} / \left| \frac{d\vec{r}}{dr} \right|, \quad \hat{e}_\theta = \frac{\partial \vec{r}}{\partial \theta} / \left| \frac{d\vec{r}}{d\theta} \right|, \quad \hat{e}_\phi = \frac{\partial \vec{r}}{\partial \phi} / \left| \frac{d\vec{r}}{d\phi} \right|$$

the transformation

$$\begin{cases} x = r \sin\theta \cos\phi \\ y = r \sin\theta \sin\phi \\ z = r \cos\theta \end{cases}$$

$$d\vec{r} = \frac{\partial \vec{r}}{\partial r} \cdot dr + \frac{\partial \vec{r}}{\partial \theta} d\theta + \frac{\partial \vec{r}}{\partial \phi} d\phi$$

or from cartesian

$$d\vec{r} = dx \hat{e}_x + dy \hat{e}_y + dz \hat{e}_z$$

$$\begin{aligned} &= \hat{e}_x \left[ \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial \phi} d\phi \right] + \hat{e}_y \left[ \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \theta} d\theta + \frac{\partial y}{\partial \phi} d\phi \right] + \hat{e}_z \left[ \frac{\partial z}{\partial r} dr + \frac{\partial z}{\partial \theta} d\theta + \frac{\partial z}{\partial \phi} d\phi \right] \\ &= \left[ \hat{e}_x \frac{\partial x}{\partial r} + \hat{e}_y \frac{\partial y}{\partial r} + \hat{e}_z \frac{\partial z}{\partial r} \right] dr + \left[ \hat{e}_x \frac{\partial x}{\partial \theta} + \hat{e}_y \frac{\partial y}{\partial \theta} + \hat{e}_z \frac{\partial z}{\partial \theta} \right] d\theta + \left[ \hat{e}_x \frac{\partial x}{\partial \phi} + \hat{e}_y \frac{\partial y}{\partial \phi} + \hat{e}_z \frac{\partial z}{\partial \phi} \right] d\phi \end{aligned}$$

$$\Rightarrow \frac{\partial \vec{r}}{\partial r} = \sin\theta \cos\phi \hat{e}_x + \sin\theta \sin\phi \hat{e}_y + \cos\theta \hat{e}_z = \hat{e}_r$$

$$\frac{\partial \vec{r}}{\partial \theta} = r \left[ +\cos\theta \cos\phi \hat{e}_x + \cos\theta \sin\phi \hat{e}_y - \sin\theta \hat{e}_z \right] = r \hat{e}_\theta$$

$$\frac{\partial \vec{r}}{\partial \phi} = r \left[ -\sin\phi \hat{e}_x + \cos\phi \hat{e}_y \right] = \frac{r \hat{e}_\phi}{\sin\theta}$$

$$\Rightarrow (\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi) = (\hat{e}_x, \hat{e}_y, \hat{e}_z) \begin{pmatrix} \sin\theta \cos\phi & \cos\theta \cos\phi & -\sin\phi \\ \sin\theta \sin\phi & \cos\theta \sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}$$

$\rightarrow$  SO(3) matrix  
 $\swarrow$  special       $\nwarrow$  orth

$$\text{or } (\hat{e}_x, \hat{e}_y, \hat{e}_z) = (\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi) \begin{pmatrix} \sin\theta \cos\phi & \sin\theta \sin\phi & \cos\theta \\ \cos\theta \cos\phi & \cos\theta \sin\phi & -\sin\phi \\ -\sin\phi & \cos\phi & 0 \end{pmatrix}$$

$$d\vec{r} = \hat{e}_r dr + \hat{e}_\theta r d\theta + \hat{e}_\phi r \sin\theta d\phi$$

$$d\hat{e}_r(r, \theta, \phi) = \frac{\partial \hat{e}_r}{\partial r} dr + \frac{\partial \hat{e}_r}{\partial \theta} d\theta + \frac{\partial \hat{e}_r}{\partial \phi} d\phi$$

$$\hat{e}_r = \sin\theta \cos\phi \hat{e}_x + \sin\theta \sin\phi \hat{e}_y + \cos\theta \hat{e}_z \Rightarrow \frac{\partial \hat{e}_r}{\partial r} = 0$$

$$\frac{\partial \hat{e}_r}{\partial \theta} = +\cos\theta \cos\phi \hat{e}_x + \cos\theta \sin\phi \hat{e}_y - \sin\theta \hat{e}_z = \hat{e}_\theta$$

$$\frac{\partial \hat{e}_r}{\partial \phi} = -\sin\theta \sin\phi \hat{e}_x + \sin\theta \cos\phi \hat{e}_y = \sin\theta \hat{e}_\phi$$

$$\Rightarrow d\hat{e}_r = \hat{e}_\theta d\theta + \hat{e}_\phi \sin\theta d\phi$$

$$\hat{e}_\theta = \cos\theta \cos\phi \hat{e}_x + \cos\theta \sin\phi \hat{e}_y - \sin\theta \hat{e}_z \Rightarrow \frac{\partial \hat{e}_\theta}{\partial r} = 0$$

$$\frac{\partial \hat{e}_\theta}{\partial \theta} = -\sin\theta \cos\phi \hat{e}_x - \sin\theta \sin\phi \hat{e}_y - \cos\theta \hat{e}_z = -\hat{e}_r$$

$$\frac{\partial \hat{e}_\theta}{\partial \phi} = -\cos\theta \sin\phi \hat{e}_x + \cos\theta \cos\phi \hat{e}_y = \cos\theta \hat{e}_\phi$$

$$\Rightarrow d\hat{e}_\theta = -\hat{e}_r d\theta + \hat{e}_\phi \cos\theta d\phi$$

$$\hat{e}_\phi = -\sin\phi \hat{e}_x + \cos\phi \hat{e}_y \Rightarrow \frac{\partial \hat{e}_\phi}{\partial r} = 0, \quad \frac{\partial \hat{e}_\phi}{\partial \theta} = 0$$

$$\frac{\partial \hat{e}_\phi}{\partial \phi} = -\cos\phi \hat{e}_x - \sin\phi \hat{e}_y = -(\hat{e}_r \sin\theta + \hat{e}_\theta \cos\theta)$$

$$\Rightarrow d\hat{e}_\phi = -(\hat{e}_r \sin\theta + \hat{e}_\theta \cos\theta) d\phi$$

$$\Rightarrow (d\hat{e}_r, d\hat{e}_\theta, d\hat{e}_\phi) = (\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi) \begin{pmatrix} 0 & -d\theta & -\sin\theta d\phi \\ d\theta & 0 & -\cos\theta d\phi \\ \sin\theta d\phi & \cos\theta d\phi & 0 \end{pmatrix}$$

\*\* metric matrix  $d\vec{r} = dr \hat{e}_r + r d\theta \hat{e}_\theta + r \sin\theta d\phi \hat{e}_\phi$

$$(dr)^2 = (dr)^2 + (r d\theta)^2 + (r \sin\theta d\phi)^2$$

generally  $(dr)^2 = g_{ij}(u^i, u^j, u^k) du^i du^j$ ,

here  $g_{ij} = \begin{pmatrix} 1 & & \\ & r^2 & \\ & & r^2 \sin^2\theta \end{pmatrix}$  . for spherical coordinates.

\*\* volume elements

$$dv = dx dy dz = \left| \frac{\partial(x, y, z)}{\partial(u^1, u^2, u^3)} \right| du^1 du^2 du^3$$

↑  
Jacobian

What's the relation between the Jacobian and metric matrix?

$$g_{ij} = \frac{\partial \vec{r}}{\partial u^i} \cdot \frac{\partial \vec{r}}{\partial u^j} = \frac{\partial x}{\partial u^i} \cdot \frac{\partial x}{\partial u^j} + \frac{\partial y}{\partial u^i} \cdot \frac{\partial y}{\partial u^j} + \frac{\partial z}{\partial u^i} \cdot \frac{\partial z}{\partial u^j}$$

$$\Rightarrow (g)_{ij} = \left\{ \begin{matrix} \left( \begin{matrix} \frac{\partial x}{\partial u^1} & \frac{\partial y}{\partial u^1} & \frac{\partial z}{\partial u^1} \\ \frac{\partial x}{\partial u^2} & \frac{\partial y}{\partial u^2} & \frac{\partial z}{\partial u^2} \\ \frac{\partial x}{\partial u^3} & \frac{\partial y}{\partial u^3} & \frac{\partial z}{\partial u^3} \end{matrix} \right) \left( \begin{matrix} \frac{\partial x}{\partial u^1} & \frac{\partial y}{\partial u^1} & \frac{\partial z}{\partial u^1} \\ \frac{\partial x}{\partial u^2} & \frac{\partial y}{\partial u^2} & \frac{\partial z}{\partial u^2} \\ \frac{\partial x}{\partial u^3} & \frac{\partial y}{\partial u^3} & \frac{\partial z}{\partial u^3} \end{matrix} \right) \end{matrix} \right\}_{ij} \Rightarrow \left| \frac{\partial(xyz)}{\partial(u^1 u^2 u^3)} \right| = \sqrt{|g|}$$

$$\Rightarrow \boxed{dv = \sqrt{|g|} du^1 du^2 du^3}$$

for spherical coordinate, we have  $|g| = r^4 \sin^2 \theta$

$$\Rightarrow \boxed{dv = r^2 \sin \theta dr d\theta d\phi}$$

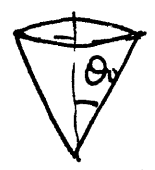
Surface integral : ① say fix  $r = R$ , we have area element sphere on the

$$d\vec{r} = R d\theta \hat{e}_\theta + R \sin \theta d\phi \hat{e}_\phi$$

$$\Rightarrow (d\vec{r})^2 = R^2 (d\theta)^2 + R^2 \sin^2 \theta (d\phi)^2 \Rightarrow g_{ij} = \begin{pmatrix} R^2 & \\ & R^2 \sin^2 \theta \end{pmatrix}$$

$$\boxed{ds = \sqrt{|g|} d\theta d\phi = R^2 \sin \theta d\theta d\phi}$$

② fix  $\theta = \theta_0 \Rightarrow d\vec{r} = dr \hat{e}_r + r \sin \theta_0 d\phi \hat{e}_\phi$



$$\Rightarrow (d\vec{r})^2 = (dr)^2 + (r \sin \theta_0)^2 (d\phi)^2 \Rightarrow g_{ij} = \begin{pmatrix} 1 & \\ & r^2 \sin^2 \theta_0 \end{pmatrix}$$

$$\Rightarrow \boxed{ds = \sqrt{|g|} dr d\phi = r \sin \theta_0 dr d\phi}$$

# \* gradient, divergence, curl in curvilinear coordinates

(5)

For orthogonal curvilinear coordinates

$$\nabla f = \sum_i \frac{1}{\sqrt{g_{ii}}} \frac{\partial f}{\partial u^i} \hat{e}_{u^i}$$

$$\nabla f(u^1, u^2, u^3) = \frac{1}{\sqrt{g_{11}}} \frac{\partial f}{\partial u^1} \hat{e}_{u^1} + \frac{1}{\sqrt{g_{22}}} \frac{\partial f}{\partial u^2} \hat{e}_{u^2} + \frac{1}{\sqrt{g_{33}}} \frac{\partial f}{\partial u^3} \hat{e}_{u^3}$$

ex: for spherical coordinates  $g_{rr} = 1, g_{\theta\theta} = r^2, g_{\phi\phi} = r^2 \sin^2 \theta$

$$\nabla f(r, \theta, \phi) = \frac{\partial f}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \hat{e}_\phi$$

for vector field  $\vec{v}(u^1, u^2, u^3) = v^1 \hat{e}_{u^1} + v^2 \hat{e}_{u^2} + v^3 \hat{e}_{u^3}$

$$\nabla \cdot \vec{v} = \frac{1}{\sqrt{g_{11} g_{22} g_{33}}} \left[ \frac{\partial (\sqrt{g_{22} g_{33}} v^1)}{\partial u^1} + \frac{\partial (\sqrt{g_{33} g_{11}} v^2)}{\partial u^2} + \frac{\partial (\sqrt{g_{11} g_{22}} v^3)}{\partial u^3} \right]$$

ex for spherical coordinates

$$\nabla \cdot \vec{v} = \frac{1}{\sqrt{\det g}} \sum_i \frac{\partial}{\partial u^i} \left[ \sqrt{\frac{\det g}{g_{ii}}} v^i \right]$$

$$\nabla \cdot \vec{v}(r, \theta, \phi) = \frac{1}{r^2 \sin \theta} \left[ \frac{\partial}{\partial r} [r^2 \sin \theta v_r] + \frac{\partial}{\partial \theta} [r \sin \theta v_\theta] + \frac{\partial}{\partial \phi} [r v_\phi] \right]$$

$$\nabla \cdot \vec{v}(r, \theta, \phi) = \frac{1}{r^2} \frac{\partial}{\partial r} [r^2 v_r] + \frac{1}{r} \frac{\partial}{\partial \theta} [\sin \theta v_\theta] + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} v_\phi$$

The Laplace operator

$$\nabla^2 f = \nabla \cdot \nabla f = \frac{1}{\sqrt{g_{11} g_{22} g_{33}}} \left[ \frac{\partial}{\partial u^1} (\sqrt{g_{22} g_{33}} \frac{\partial f}{\partial u^1}) + \frac{\partial}{\partial u^2} (\sqrt{g_{33} g_{11}} \frac{\partial f}{\partial u^2}) + \frac{\partial}{\partial u^3} (\sqrt{g_{11} g_{22}} \frac{\partial f}{\partial u^3}) \right]$$

$$v^1 = \frac{1}{\sqrt{g_{11}}} \frac{\partial f}{\partial u^1}, \quad v^2 = \frac{1}{\sqrt{g_{22}}} \frac{\partial f}{\partial u^2}, \quad v^3 = \frac{1}{\sqrt{g_{33}}} \frac{\partial f}{\partial u^3}$$

$$\Rightarrow \nabla^2 f = \frac{1}{\sqrt{g_{11} g_{22} g_{33}}} \left[ \frac{\partial}{\partial u^1} \left( \sqrt{\frac{g_{22} g_{33}}{g_{11}}} \frac{\partial f}{\partial u^1} \right) + \frac{\partial}{\partial u^2} \left( \sqrt{\frac{g_{33} g_{11}}{g_{22}}} \frac{\partial f}{\partial u^2} \right) + \frac{\partial}{\partial u^3} \left( \sqrt{\frac{g_{11} g_{22}}{g_{33}}} \frac{\partial f}{\partial u^3} \right) \right]$$

$$\vec{\nabla} f = \frac{1}{\sqrt{\det g}} \sum_i \frac{\partial}{\partial u^i} \left[ \frac{\sqrt{\det g}}{g_{ii}} \frac{\partial f}{\partial u^i} \right]$$

For spherical coordinates  $\sqrt{\det g} = r^2 \sin \theta$

$$g_{rr} = 1$$

$$g_{\theta\theta} = r^2$$

$$g_{\phi\phi} = r^2 \sin^2 \theta$$

$$\vec{\nabla} f = \frac{1}{r^2 \sin \theta} \left[ \frac{\partial}{\partial r} (r^2 \sin \theta \frac{\partial f}{\partial r}) + \frac{\partial}{\partial \theta} \left[ \sin \theta \frac{\partial f}{\partial \theta} \right] + \frac{\partial}{\partial \phi} \left[ \frac{1}{\sin \theta} \frac{\partial f}{\partial \phi} \right] \right]$$

$$\vec{\nabla} f = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial f}{\partial r}) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left[ \sin \theta \frac{\partial f}{\partial \theta} \right] + \frac{1}{r^2 \sin \theta} \frac{\partial^2 f}{\partial \phi^2}$$

curl: for  $\vec{v} = v^1 \hat{e}_1 + v^2 \hat{e}_2 + v^3 \hat{e}_3$

define  $\nabla \times \vec{v} = (\nabla \times \vec{v})^1 \hat{e}_1 + (\nabla \times \vec{v})^2 \hat{e}_2 + (\nabla \times \vec{v})^3 \hat{e}_3$

then  $(\nabla \times \vec{v})^1 = \frac{\sqrt{g_{11}}}{\sqrt{\det g}} \left[ \frac{\partial}{\partial u^2} (\sqrt{g_{33}} u^3) - \frac{\partial}{\partial u^3} (\sqrt{g_{22}} u^2) \right]$

$$(\nabla \times \vec{v})^2 = \frac{\sqrt{g_{22}}}{\sqrt{\det g}} \left[ \frac{\partial}{\partial u^3} (\sqrt{g_{11}} u^1) - \frac{\partial}{\partial u^1} (\sqrt{g_{33}} u^3) \right]$$

$$(\nabla \times \vec{v})^3 = \frac{\sqrt{g_{33}}}{\sqrt{\det g}} \left[ \frac{\partial}{\partial u^1} (\sqrt{g_{22}} u^2) - \frac{\partial}{\partial u^2} (\sqrt{g_{11}} u^1) \right]$$

or  $(\nabla \times \vec{v})^i = \frac{1}{\sqrt{\det g}} \epsilon_{ijk} \sqrt{g_{ii}} \left[ \frac{\partial}{\partial u^j} (\sqrt{g_{kk}} u^k) - \frac{\partial}{\partial u^k} (\sqrt{g_{jj}} u^j) \right]$

⑦

for spherical coordinate  $\sqrt{\det g} = r^2 \sin \theta$ ,  $\sqrt{g_{rr}} = 1$ ,  $\sqrt{g_{\theta\theta}} = r$ ,  $\sqrt{g_{\varphi\varphi}} = r \sin \theta$

$$\Rightarrow (\nabla \times \vec{v})^r = \frac{1}{r^2 \sin \theta} \left[ \frac{\partial}{\partial \theta} [r \sin \theta v_\varphi] - \frac{\partial}{\partial \varphi} [r v_\theta] \right] = \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} (\sin \theta v_\varphi) - \frac{\partial v_\theta}{\partial \varphi} \right]$$

$$(\nabla \times \vec{v})^\theta = \frac{r}{r^2 \sin \theta} \left[ \frac{\partial}{\partial \varphi} [v_r] - \frac{\partial}{\partial r} [r \sin \theta v_\varphi] \right] = \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \varphi} v_r \right] - \frac{1}{r} \frac{\partial}{\partial r} (r v_\varphi)$$

$$(\nabla \times \vec{v})^\varphi = \frac{r \sin \theta}{r^2 \sin \theta} \left[ \frac{\partial}{\partial r} [r v_\theta] - \frac{\partial}{\partial \theta} [v_r] \right] = \frac{1}{r} \left[ \frac{\partial}{\partial r} (r v_\theta) - \frac{\partial v_r}{\partial \theta} \right]$$

• exercise: cylindrical coordinate

$$\begin{aligned} d\vec{r} &= dr \hat{e}_r + r d\theta \hat{e}_\theta + dz \hat{e}_z \\ |d\vec{r}|^2 &= (dr)^2 + (r d\theta)^2 + (dz)^2 \Rightarrow g_{rr}=1 \quad g_{\theta\theta}=r^2, \quad g_{zz}=1 \\ \sqrt{\det g} &= r \end{aligned}$$

$$\Rightarrow \nabla f(r, \theta, z) = \frac{\partial}{\partial r} f \hat{e}_r + \frac{1}{r} \frac{\partial}{\partial \theta} f \hat{e}_\theta + \frac{\partial}{\partial z} f \hat{e}_z$$

$$\begin{aligned} \nabla \cdot \vec{v}(r, \theta, z) &= \frac{1}{r} \left\{ \frac{\partial}{\partial r} [r v_r] + \frac{\partial}{\partial \theta} [v^\theta] + \frac{\partial}{\partial z} [r v^z] \right\} \\ &= \frac{1}{r} \frac{\partial}{\partial r} [r v_r] + \frac{1}{r} \frac{\partial}{\partial \theta} v^\theta + \frac{\partial}{\partial z} v^z \end{aligned}$$

$$\begin{aligned} \nabla^2 f(r, \theta, z) &= \frac{1}{r} \left[ \frac{\partial}{\partial r} \left[ r \frac{\partial}{\partial r} f \right] + \frac{\partial}{\partial \theta} \left( \frac{r}{r^2} \frac{\partial}{\partial \theta} f \right) + \frac{\partial}{\partial z} \left( r \frac{\partial}{\partial z} f \right) \right] \\ &= \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} f \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} f + \frac{\partial^2}{\partial z^2} f \end{aligned}$$

$$(\nabla \times \vec{v})^r = \frac{1}{r} \left( \frac{\partial}{\partial \theta} v_z - \frac{\partial}{\partial z} r v_\theta \right) = \frac{1}{r} \frac{\partial}{\partial \theta} v_z - \frac{\partial}{\partial z} v_\theta$$

$$(\nabla \times \vec{v})^\theta = \frac{r}{r} \left[ \frac{\partial}{\partial z} v_r - \frac{\partial}{\partial r} v_z \right] = \frac{\partial v_r}{\partial z} - \frac{\partial}{\partial r} v_z$$

$$(\nabla \times \vec{v})^z = \frac{1}{r} \left[ \frac{\partial}{\partial r} (r v_\theta) - \frac{\partial}{\partial \theta} v_r \right] = \frac{1}{r} \left[ \frac{\partial}{\partial r} (r v_\theta) - \frac{\partial}{\partial \theta} v_r \right]$$