Lect 2 Differential calculus

§ gradient:

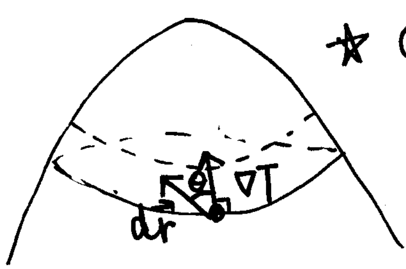
Consider a scalar function $T(x, y, z)$.

$$dT(x,y,z) = \frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial y} dy + \frac{\partial T}{\partial z} dz$$

$$= \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial y}\hat{z}\right)\cdot\left(dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}\right)$$

$$= \nabla T \cdot d\vec{r}\ , \text{ where } \nabla T = \frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}$$

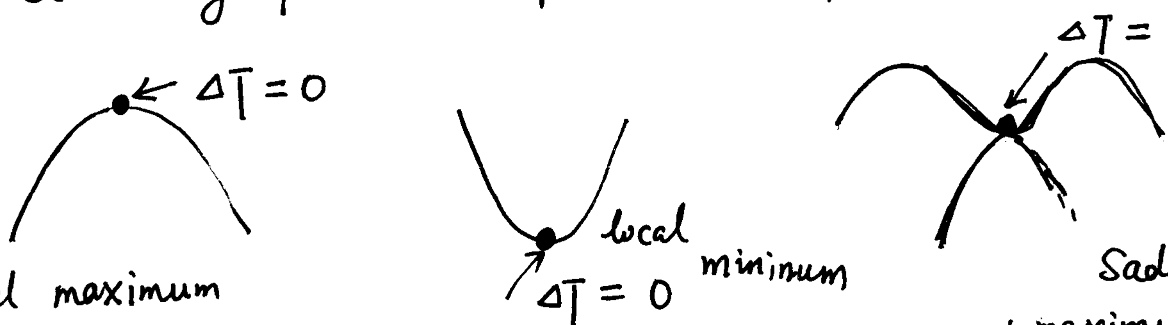
$\Rightarrow$ $dT = |\nabla T|\cdot|d\vec{r}|\cos\theta$, where θ is the angle between ∇T and $d\vec{r}$.



★ Climb a hill. The direction of ∇h is the steepest direction.
$h(x, y, z)$

The magnitude of ∇h is the slope along this hardest direction.

• Stationary point: $\nabla T = 0$. $dT \propto O(\Delta r)^2$



local maximum

local minimum

Saddle point
: maximum along one direction
minimum along another direction

Ex: $r = \sqrt{x^2+y^2+z^2}$

$$\nabla r = \frac{\partial r}{\partial x}\hat{x} + \frac{\partial r}{\partial y}\hat{y} + \frac{\partial r}{\partial z}\hat{z} = \frac{1}{\sqrt{x^2+y^2+z^2}}(x\hat{x} + y\hat{y} + z\hat{z})$$

$$= \frac{\vec{r}}{r} = \hat{r}$$

→ nabla operator: $\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial y}$ (vector operator)

The effect of ∇ is: for any scalar function $f(x,y,z)$

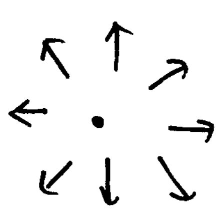
$$\boxed{\nabla f = \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z}}$$

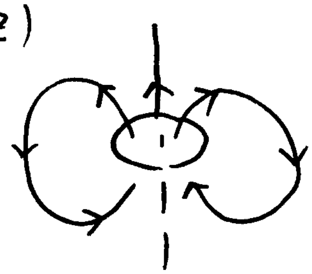
§ divergence: — applying the ∇-operator on a vector function: the point product

for a vector field $\vec{v} = v_x(x,y,z)\,\hat{x} + v_y(x,y,z)\,\hat{y} + v_z(x,y,z)\,\hat{z}$ each component is a function of (x,y,z)

Example: 1 electric field $\vec{E}(x,y,z) = [E_x(x,y,z),\ E_y(x,y,z),\ E_z(x,y,z)]$

2 $\vec{B}(x,y,z)$





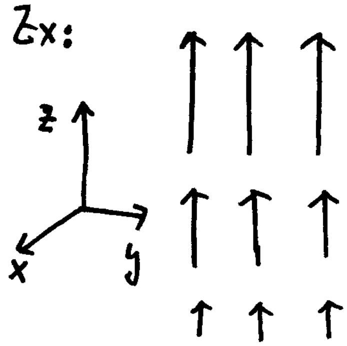
divergence $\quad \nabla \cdot \vec{v} = \left(\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) \left(v_x \hat{x} + v_y \hat{y} + v_z \hat{z} \right)$

$$= \frac{\partial}{\partial x} v_x + \frac{\partial}{\partial y} v_y + \frac{\partial}{\partial z} v_z$$

* a scalar function has no divergence.

* Physical meaning: $\nabla \cdot \vec{v}$ is a measure of the vecter $\vec{v}$ spreads out from a point ~~of~~.

Ex:



$\vec{v} = (0,\ 0,\ v_z(z))$

$\nabla \cdot \vec{v} = \frac{\partial}{\partial z} v_z > 0$.

Consider a small box:

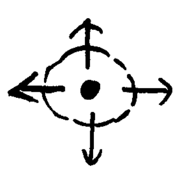
$\underbrace{\text{The flux}}_{\text{net}}$ of $\vec{v}$ to outside

$$v_z(z+\Delta z)\,\Delta y \Delta x - v_z(z)\,\Delta y \Delta x$$

$$= \frac{\partial}{\partial z} v_z\, \Delta x \Delta y \Delta z$$

$$= \nabla \cdot \vec{v}\ (dx\,dy\,dz)$$

or Ex



$\vec{v} = (x, y, z) = \vec{r}$

Consider a small sphere with Δr

the flux goes outsiden $\quad 4\pi(\Delta r)^2 \cdot \Delta r = 4\pi (\Delta r)^3$

$$= \frac{4}{3}\pi (\Delta r)^3 \cdot (\nabla \cdot \vec{r})$$

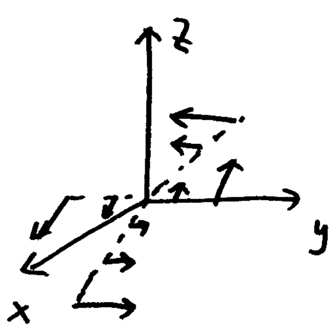
$$= \Delta V\ (\nabla \vec{v})$$

§3. curl

$$\nabla \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} = \hat{x}\left(\frac{\partial}{\partial y} v_z - \frac{\partial}{\partial z} v_y\right) + \hat{y}\left(\frac{\partial}{\partial z} v_x - \frac{\partial}{\partial x} v_z\right) + \hat{z}\left(\frac{\partial}{\partial x} v_y - \frac{\partial}{\partial y} v_x\right)$$

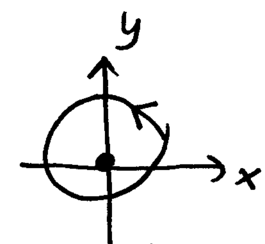
: $\nabla \times \vec{v}$ is a measure of how much $\vec{v}$ "curls around" the point.

The direction of $\nabla \times \vec{v}$ is perpendicular to the circulation plane



Ex: $\vec{v} = (-y,\ x,\ 0)$

circulation around a circle of radial Δr, along the counter-clockwise direction



$$\Delta r \cdot 2\pi \Delta r = \pi (\Delta r)^2 \cdot 2$$

$$\nabla \times \vec{v} = \hat{z} \cdot 2$$

$$\Rightarrow \text{circulation} = \Delta S\, \hat{z} \cdot (\nabla \times \vec{v})$$

§ Product rules

$$\left\{\begin{array}{l}\nabla(f+g) = \nabla f + \nabla g \\ \nabla\cdot(\vec{A}+\vec{B}) = \nabla\vec{A} + \nabla\vec{B} \\ \nabla\times(\vec{A}+\vec{B}) = \nabla\times\vec{A} + \nabla\times\vec{B}\end{array}\right.$$

$$\nabla(kf) = k\nabla f$$

$$\nabla\cdot(k\vec{A}) = k\nabla\cdot\vec{A}$$

$$\nabla\times(k\vec{A}) = k\nabla\times\vec{A}$$

k is a cnst

linear opearater

properties of

more

$$\nabla(fg) = f\nabla g + \nabla f\, g$$

$$\nabla(f/g) = \frac{1}{g}\nabla f - \frac{f\nabla g}{g^2}$$

$$\nabla\cdot(f\vec{A}) = f(\nabla\cdot\vec{A}) + \vec{A}\cdot\nabla f$$

Not required to remember

$$\left\{\begin{array}{l}\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B}) \\ \nabla\times(f\vec{A}) = f(\nabla\times\vec{A}) - \vec{A}\times(\nabla f) \\ \nabla\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\nabla)\vec{A} - (\vec{A}\cdot\vec{\nabla})\vec{B} + \vec{A}(\nabla\cdot\vec{B}) - \vec{B}(\nabla\cdot\vec{A})\end{array}\right.$$

§: • Laplacian : divergence of gradient

$$\nabla\cdot(\nabla T) = \frac{\partial}{\partial x}(\nabla T)_x + \frac{\partial}{\partial y}(\nabla T)_y + \frac{\partial}{\partial z}(\nabla T)_z$$

$$= \frac{\partial^2}{\partial x^2}T + \frac{\partial^2}{\partial y^2}T + \frac{\partial^2}{\partial z^2}T = \nabla^2 T$$

• curl of gradient is zero

$$\nabla\times(\nabla T) = 0$$

$$[\nabla\times(\nabla T)]_z = \frac{\partial}{\partial x}(\nabla T)_y - \frac{\partial}{\partial y}(\nabla T)_x = \left(\frac{\partial}{\partial x}\frac{\partial}{\partial y} - \frac{\partial}{\partial y}\frac{\partial}{\partial x}\right)T = 0$$

similarly $[\nabla\times(\nabla T)]_x = [\nabla\times(\nabla T)]_y = 0.$

• divergence of a curl is zero

$$\nabla\cdot(\nabla\times\vec{v}) = 0$$

$$\nabla\cdot(\nabla\times\vec{v}) = \frac{\partial}{\partial x}[\nabla\times\vec{v}]_x + \frac{\partial}{\partial y}(\nabla\times\vec{v})_y + \frac{\partial}{\partial z}(\nabla\times\vec{v})_z$$

$$= \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}v_z - \frac{\partial}{\partial z}v_y\right] + \frac{\partial}{\partial y}\left[\frac{\partial}{\partial z}v_x - \frac{\partial}{\partial x}v_z\right] + \frac{\partial}{\partial z}\left[\frac{\partial}{\partial x}v_y - \frac{\partial}{\partial y}v_x\right]$$

$$= \left(\frac{\partial}{\partial x}\frac{\partial}{\partial y} - \frac{\partial}{\partial y}\frac{\partial}{\partial x}\right)v_z + \left(\frac{\partial}{\partial z}\frac{\partial}{\partial x} - \frac{\partial}{\partial x}\frac{\partial}{\partial z}\right]v_y + \left[\frac{\partial}{\partial y}\frac{\partial}{\partial z} - \frac{\partial}{\partial z}\frac{\partial}{\partial y}\right]v_x$$

$$= 0$$

- Curl of curl

$$\nabla \times (\nabla \times \vec{v}) = \nabla(\nabla \cdot \vec{v}) - \nabla^2 \vec{v}$$ ← leave as an excercise

definition of $\nabla^2 v^2 = (\nabla^2 v_x,\ \nabla^2 v_y,\ \nabla^2 v_z)$