Perturbations to the Lunar Orbit

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Abstract

In this paper, a general approach to performing perturbation analysis on a two-dimensional orbit is presented. The specific examples of the solar tidal and the gravitomagnetic influences on the lunar orbit are developed here. In these examples, only periodic behaviors are kept, as these are the signals readily measured by ranging techniques.

1 The Nominal Orbit

In polar coordinates, the Lagrangian for a point mass, m, moving around a central (fixed) mass, M, is developed as follows:

$$L = T - U = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\omega^2 + \frac{GMm}{r},$$

where ω is the angular velocity of the orbiting body. From this Lagrangian, we derive the following equation of motion, in which the mass of the orbiting body has been canceled:

$$\ddot{r} = -\frac{GM}{r^2} + r\omega^2 = a(r) + \frac{l^2}{r^3}, \tag{1}$$

where $a(r)$ is the central acceleration, and $l = r^2\omega$ is the (specific) angular momentum. In a circular orbit, $\ddot{r} = 0$, so $r^3\omega^2 = GM$, which recovers Kepler's third law.

2 Perturbation

If $r(t)$ is a solution that satisfies Eq. 1, then we can explore the effect that an acceleration perturbation, $\vec{\delta a}$, has on the solution. We let $r(t) \rightarrow r(t) + \delta r(t)$. Meanwhile, the angular momentum must be allowed to change, so we let $l(t) \rightarrow l(t) + \delta l(t)$. The acceleration perturbation can be decomposed into a radial part and a tangential part, δa_r, and δa_τ, respectively. In addition to Eq. 1, we need to know that

$$\dot{l} = r^2\dot{\omega} = r\delta a_\tau. \tag{2}$$

Placing the perturbed values of $r(t)$ and $l(t)$ into Eq. 1, we have

$$\ddot{r} + \ddot{\delta r} = -\frac{GM}{(r+\delta r)^2} + \frac{(l+\delta l)^2}{(r+\delta r)^3} + \delta a_r. \tag{3}$$

We have added δa_r directly to this relation, as the first term on the right-hand side of Eq. 3 is the radial acceleration, $a(r)$, which is perturbed by the amount δa_r. We have not yet accounted for δa_τ, as this is still hidden in δl. Expanding Eq. 3, and keeping only the first order terms in $\delta r/r$ and $\delta l/l$, we see that Eq. 1 is exactly replicated. Since $r(t)$ is already a valid solution to Eq. 1, this part subtracts away, leaving only

$$\ddot{\delta r} = -\frac{2GM}{r^3}\delta r - \frac{3l^2}{r^4}\delta r + \frac{2l}{r^3}\delta l + \delta a_r. \tag{4}$$

We can replace δl with

$$\delta l = \int \dot{l} dt' = \int r^2 \dot{\omega} dt' = r \int \delta a_\tau dt',$$

where t' is a dummy time variable over which to integrate the perturbation. Using Kepler's relation, along with the definition of l, we can reduce Eq. 4 to

$$\begin{aligned} \ddot{\delta r} &= 2\omega^2 \delta r - 3\omega^2 \delta r + 2\tfrac{\omega}{r} \delta l + \delta a_r \\ &= -\omega^2 \delta r + \delta a_r + 2\omega \textstyle\int \delta a_\tau dt' \end{aligned}$$

leaving us with the differential equation for δr that looks like

$$\ddot{\delta r} + \omega^2 \delta r = \delta a_r + 2\omega \int \delta a_\tau dt'. \tag{5}$$

This is a simple forced harmonic oscillator. In the absence of a perturbing acceleration, $\vec{\delta a}$, this would result in a general solution that oscillates at the resonant orbital frequency, ω. Thus it describes simple elliptical oscillation about the nominal orbit—our original $r(t)$ that satisfied Eq. 1.

Given functional forms for δa_r and δa_τ, one can solve Eq. 5 for δr to understand the range variation one would see under such an influence. This procedure is clarified through the following example of the gravitomagnetic perturbation.

3 Solar Tidal Perturbation

What influence does the sun have on the geocentric lunar orbit? The dominant influence is tidal, resulting from the differential pull of the sun on the earth and moon as the moon ventures away from the nominal earth orbit. Specifically, we can express the accelerations of the earth and moon toward the sun, the difference being the net tidal acceleration on the moon, which becomes our $\vec{\delta a}$ perturbation:

$$\mathbf{a}_e = -\frac{GM}{d^2}\hat{\mathbf{j}}$$

is the acceleration on the earth, where M is the solar mass, d is the earth-sun distance, and the coordinate system is arbitrarily chosen to have the sun in the $-\hat{\mathbf{j}}$ direction. The acceleration on the moon is

$$\mathbf{a}_m = -\frac{GM}{\rho^2}\left(\frac{a\sin D}{\rho}\hat{\mathbf{i}} + \frac{d - a\cos D}{\rho}\hat{\mathbf{j}}\right),$$

where D is the synodic phase angle between the sun and the moon as seen from the earth, a is the nominal (circular) earth-moon distance, and ρ is the distance between the moon and sun. The net acceleration is then

$$\mathbf{a_{net}} = \mathbf{a}_m - \mathbf{a}_e = -\frac{GMa\sin D}{\rho^3}\hat{\mathbf{i}} + \left(\frac{GM}{d^2} - \frac{GM}{\rho^2}\frac{d - a\cos D}{\rho}\right)\hat{\mathbf{j}}.$$

We need the radial and tangential components of $\vec{\delta a}$ for inclusion into Eq. 5, which we obtain by dotting $\vec{\delta a}$ with $\hat{\mathbf{r}} = \sin D\hat{\mathbf{i}} - \cos D\hat{\mathbf{j}}$ and $\hat{\tau} = \cos D\hat{\mathbf{i}} + \sin D\hat{\mathbf{j}}$. This works out to

$$\delta a_r = \frac{GM}{\rho^3}(d\cos D - a) - \frac{GM}{d^2}\cos D,$$

and

$$\delta a_\tau = \frac{GM}{d^2}\sin D - \frac{GMd}{\rho^3}\sin D.$$

Now the moon-sun distance, ρ, can be obtained from the geometry as $\rho^2 = d^2 + a^2 - 2ad\cos D$. Note that since a is about 400 times smaller than d, we can approximate ρ to first order in this quantity without sacrificing much precision. Specifically, we care about ρ^{-3}:

$$\rho^{-3} = (d^2 + a^2 - 2ad\cos D)^{-\frac{3}{2}} = d^{-3}(1 + x^2 - 2x\cos D)^{-\frac{3}{2}} \approx d^{-3}(1 + 3\frac{a}{d}\cos D),$$

where $x \equiv a/d$ is the small quantity for which second-order terms may be ignored. Now we have

$$\delta a_r \approx -\frac{GMa}{d^3}(1 - 3\cos^2 D + 3\frac{a}{d}\cos D) \approx \frac{GMa}{2d^3}(1 + 3\cos 2D), \tag{6}$$

where we have ignored the small a/d cosine term, and have expressed the squared cosine by its double angle equivalent. Likewise,

$$\delta a_\tau \approx -\frac{3GMa}{2d^3}\sin 2D. \tag{7}$$

These acceleration perturbations in hand, we are ready to use Eq. 5 to solve the system. First, we integrate the δa_τ term. Noting that the rate at which D advances is $\dot{D} = \omega - \Omega$, the difference between the lunar orbital frequency and the earth's orbital frequency, we can construct an arbitrary $2D$ argument as $[2(\omega - \Omega)t' + \phi]$, where ϕ is an arbitrary phase depending on the choice of $t' = 0$. The integral (without the numerical pre-factor) is then

$$2\omega \int_{t_0}^{t} \sin[2(\omega - \Omega)t' + \phi]dt' = -\frac{\omega}{\omega - \Omega}\cos 2D + \text{const.} \tag{8}$$

We have buried any initial phase in the integration constant, effectively defining t so that $D = (\omega - \Omega)t$ without a phase correction—setting $t = 0$ to be coincident with $D = 0$. Thus the integration constant essentially represents the integral from $t = t_0$ to $t = 0$. Now placing Eq. 6 and Eq. 8 into Eq. 5 using the appropriate pre-factor from Eq. 7, we have

$$\ddot{\delta r} + \omega^2 \delta r = \frac{GMa}{2d^3}\left[1 + 3\cos 2D + 3\frac{\omega}{\omega - \Omega}\cos 2D + \text{const.}\right]. \tag{9}$$

Notice on the right-hand-side of Eq. 9 there are both periodic and constant terms. The effect of the constant terms is to rescale the orbit, such that a re-definition of ω and the nominal radius can absorb this constant. In other words, if $r(t)$ is a solution to Eq. 1, then the constant terms act to make $r(t) + \text{const.}$ the new solution, with ω modified accordingly. So for our purposes, we will ignore these terms so that we may develop only the periodic response. The remaining differential equation for δr is

$$\ddot{\delta r} + \omega^2 \delta r = \frac{3GMa}{2d^3}\frac{2\omega - \Omega}{\omega - \Omega}\cos 2D. \tag{10}$$

It is clear that the solution to this equation must be $\delta r = \alpha \cos 2D$, and all that remains is to solve for the amplitude, α. Again setting $2D = 2(\omega - \Omega)t$, placing our expression for δr into Eq. 10 results in

$$-4(\omega - \Omega)^2 \alpha \cos 2D + \omega^2 \alpha \cos 2D = \frac{3GMa}{2d^3}\frac{2\omega - \Omega}{\omega - \Omega}\cos 2D,$$

so that

$$\alpha = \frac{3GMa}{2d^3}\frac{2\omega - \Omega}{\omega - \Omega}\frac{1}{\omega^2 - (\omega - \Omega)^2}.$$

Using Kepler's relation for the Earth, namely $\Omega^2 d^3 = GM$, and eliminating second-order terms in Ω/ω in the denominator, we get the tidal range variation solution to be

$$\delta r = -\frac{3}{2}a\left(\frac{\Omega}{\omega}\right)^2 \frac{2 - \eta}{1 - \eta}\frac{1}{3 - 8\eta}\cos 2D \approx -2842\cos 2D\,\text{km}, \tag{11}$$

where $\eta \equiv \Omega/\omega \approx 0.075$ is used for the sake of tidiness. The numerical result very closely matches the "true" $\cos 2D$ perturbation of the lunar orbit of -2996 km. Note that the factors with η would reduce to 0.667 if η is assumed to be small and therefore neglected, but computes to 0.86 when considered: 30% larger than the gross estimate.

4 Gravitomagnetic Perturbation

The gravitomagnetic acceleration of body i due to bodies j is given by

$$\mathbf{a}_i = \sum_j \frac{\mu_j(2+2\gamma)}{c^2 r_{ij}^3}\mathbf{v}_i \times (\mathbf{v}_j \times \mathbf{r}_{ij}) \tag{12}$$

where $\mathbf{v}_i$ and $\mathbf{v}_j$ are the velocities of bodies i and j in some arbitrary coordinate system. The vector $\mathbf{r}_{ij}$, when combined with the fraction μ_j/r_{ij}^3 constitutes the Newtonian gravitational acceleration of mass i toward mass j. The factor γ is a PPN parameter quantifying the amount of spacetime curvature produced by unit mass. In General Relativity, $\gamma = 1$. For the purposes of this development, we will express Eq. 12 differently by using the vector identity: $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$, and taking only the influence of the earth on the moon so that we have

$$\mathbf{a}_m = \frac{\mu_e(2+2\gamma)}{c^2 a^2}\left[\hat{\mathbf{r}}(\mathbf{v}_m \cdot \mathbf{v}_e) - \mathbf{v}_e(\mathbf{v}_m \cdot \hat{\mathbf{r}})\right], \tag{13}$$

where $\hat{\mathbf{r}}$ is the unit vector from the earth to the moon, and a is the earth-moon distance. We can represent the earth's velocity around the sun as $\mathbf{V}$ and the moon's velocity in the same coordinate system as $\mathbf{V}+\mathbf{u}$, where u is approximately thirty times smaller than V in magnitude. Eq. 13 then becomes

$$\vec{\delta a} = \frac{(2+2\gamma)GM}{c^2 a^2}\left[\hat{\mathbf{r}}(V^2 + \mathbf{V} \cdot \mathbf{u}) - \mathbf{V}(\mathbf{V} \cdot \hat{\mathbf{r}} + \mathbf{u} \cdot \hat{\mathbf{r}})\right]. \tag{14}$$

Note that $\mathbf{u}$ represents the geocentrically-viewed orbital velocity of the moon around the earth. Thus under the assumption of a circular orbit (about which we examine the perturbation), $\mathbf{u} \cdot \hat{\mathbf{r}} = 0$. Likewise, if we define $\hat{\tau}$ to be the tangential orbit vector at the moon that is perpendicular to $\hat{\mathbf{r}}$, $\mathbf{u} \cdot \hat{\tau} = u$. Under the assumption that the earth is in a circular orbit about the sun, the relationship between $\mathbf{V}$ (perpendicular to earth-sun line) and $\hat{\mathbf{r}}$ and $\hat{\tau}$ picks out the synodic phase angle, D. Specifically, $\mathbf{V} \cdot \hat{\mathbf{r}} = -V \sin D$ and $\mathbf{V} \cdot \hat{\tau} = -V \cos D$. Similarly, $\mathbf{V} \cdot \mathbf{u} = -Vu \cos D$. Now we can pick out the components of $\vec{\delta a}$ in the radial and tangential directions by dotting with $\hat{\mathbf{r}}$ and $\hat{\tau}$, respectively. For now, leaving off the pre-factor from 14 and dealing only with the vector math:

$$\delta a_r \propto V^2 - Vu\cos D - (-V\sin D)^2 = V^2\cos^2 D - Vu\cos D = \frac{1}{2}V^2 + \frac{1}{2}V^2\cos 2D - Vu\cos D,$$

and

$$\delta a_\tau \propto -(-V\cos D)(-V\sin D) = -\frac{1}{2}V^2\sin 2D.$$

These two break into two categories: V^2 terms that have $2D$ angular dependence, and Vu terms that have D angular dependence. We can treat each separately in solving Eq. 5. There is also a constant term in the expression for δa_r. Just as in the tidal perturbation example (discussion following Eq. 9), we can ignore the constant term since it only acts to rescale the orbit in a non-periodic way. We will deal first with the $2D$ terms, then look at the D terms.

The integration of the δa_τ term follows a similar line to that seen in Eq. 8, including arguments about the resulting constant term, which we ignore. The differential equation becomes

$$\ddot{\delta r} + \omega^2\delta r = \frac{(1+\gamma)GM}{a^2c^2}V^2\left[\cos 2D + \frac{\omega}{\omega-\Omega}\cos 2D\right] = \frac{(1+\gamma)GM}{a^2}\frac{V^2}{c^2}\frac{2\omega-\Omega}{\omega-\Omega}\cos 2D.$$

Following steps similar to those following Eq. 10, we arrive at the solution

$$\delta r \approx -(1+\gamma)\frac{V^2}{c^2}\frac{2-\eta}{1-\eta}\frac{a}{3-8\eta}\cos 2D \approx -6.54\cos 2D\,\mathrm{m},$$

where we have again made use of Kepler's relation ($\omega^2 a^3 = GM$) and have defined $\eta \equiv \Omega/\omega$. We have also rejected terms to second order in η.

The term proportional to Vu has no tangential part, so we immediately write the differential equation as

$$\ddot{\delta r} + \omega^2 \delta r = -\frac{(2+2\gamma)GM}{c^2 a^2} V u \cos D. \tag{15}$$

The solution is of the form $\delta r = \alpha \cos D$. Putting this into Eq. 15 determines α, so that the solution becomes

$$\delta r = -\frac{(2+2\gamma)GM}{c^2 a^2} \frac{Vu}{\omega^2 - (\omega - \Omega)^2} \cos D \approx -(1+\gamma)\frac{Vu}{c^2}\frac{\omega}{\Omega} a \cos D \approx -3.44 \cos D \,\mathrm{m}.$$

But the resonance produced by the interaction of synodic perturbations and the solar tidal distortion results in an amplification of $\cos D$ terms by the factor

$$Q_{\mathrm{res}} = \frac{1}{1 - 7\frac{\Omega}{\omega}} \approx 2.11$$

so that the corrected range oscillation is

$$\delta r \approx -(1+\gamma)\frac{Vu}{c^2}\frac{\omega}{\Omega}\frac{a}{1-7\eta} \cos D \approx -7.29 \cos D \,\mathrm{m}.$$

These numbers slightly disagree with the analysis in Nordtvedt's publications. Part of this is a more thorough accounting of the Ω/ω corrections. We see that such care gets us much closer to the "right" answer for the main solar tidal perturbation (Eq. 11)—we would have been about 35% shy without this. The disagreement in the $\cos D$ term is not attributable to this, but is perhaps the result of greater care in the numerical inputs. I have no way to confirm the resonance amplification factor at this time, however. Therefore the results:

$$\begin{aligned} \delta r_{2D} &= -6.54 \cos 2D \,\mathrm{m} \\ \delta r_D &= -7.29 \cos D \,\mathrm{m} \end{aligned}$$

represent the perhaps the best approximations to the gravitomagnetic range signals.

5 General Solar-Directed Perturbation

If the perturbation, $\vec{\delta a}$, is directed toward the sun, then $\delta a_r = \delta a \cos D$, and $\delta a_\tau = -\delta a \sin D$. The integral of the tangent part—dropping the constant part in the now-familiar way—is

$$2\omega \int \delta a_\tau dt' = \frac{2\omega}{\omega - \Omega} \delta a \cos D.$$

The equation for δr is then

$$\ddot{\delta r} + \omega^2 \delta r = \delta a \left(1 + \frac{2\omega}{\omega - \Omega}\right) \cos D = \delta a \frac{3\omega - \Omega}{\omega - \Omega} \cos D,$$

so that the solution for δr is

$$\delta r = \frac{3\omega - \Omega}{\omega - \Omega} \frac{\delta a}{\omega^2 - (\omega - \Omega)^2} \cos D \approx \frac{\delta a}{\omega^2} \frac{\omega}{2\Omega} \frac{3 - \eta}{1 - \eta} \cos D,$$

where we have again used η to denote Ω/ω. Numerically, this works out to $3.01 \times 10^{12} \delta a$ m, if δa is in m s^{-2}. Expressed as a fraction of the solar acceleration (5.94×10^{-3} m s^{-2}), this is $1.79 \times 10^{10} \delta a / a$. Nordtvedt's published work sets this numerical factor at 2.9×10^{10}, the difference attributable to the resonance factor (in this case, apparently about 1.62).