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Dynamics of a globally coupled laser model

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We study analytically and numerically a model of N identical globally coupled lasers. We extend the discussion of the linear stability, presented in a recent paper [Silber *et al.*, J. Opt. Soc. Am. B **10**, 1121 (1993)] for general splay states. For semiconductor lasers, we find that the splay state is neutrally stable only for a narrow range Δ, a parameter characterizing the splay state. We draw the phase diagram for semiconductor lasers using characteristic parameter values. We then analyze the model numerically, both for semiconductor lasers and for solid-state lasers. We also investigate the diffusion of the splay state on the solution branch in the presence of a noise term. Finally, we discuss the linear stability of the periodic two-cluster state found for semiconductor lasers.

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I. INTRODUCTION

The dynamics of globally coupled oscillators has attracted wide attention in recent years [1–15]. Examples of such systems can be found in biology (the synchronization of fireflies [3]) and in physics (Josephson junctions [4], coupled laser systems [5], and coupled maps [6]). Although the uncoupled oscillators typically have simple dynamics, the behavior of the coupled system is often surprisingly complex. It can consist of entrainment (the synchronization of the individual oscillators), chaos, and clustering. Much of the current research focuses on how such complicated dynamics can arise through global coupling.

In a recent paper, Silber *et al.* [7] discussed the linear stability of an array of N globally coupled lasers. In their paper, they examined three states with simple dynamics: the off state (the nonlasing state), the in-phase state (the state with perfect synchronization), and the splay state (a state where the average vanishes; see below). In this paper we investigate the model of Silber *et al.* in more detail. The analysis is similar to previous work on the complex Ginzburg-Landau equation [8,9]. We investigate solid-state lasers and semiconductor lasers, but focus primarily on the latter, since they exhibit a more diverse dynamical behavior.

Section II reviews the results obtained by Silber *et al.* We extend the linear stability analysis for the more general splay state (defined below). We plot the phase diagram for characteristic parameter values corresponding to semiconductor lasers. In Sec. III, we discuss our numerical results for solid-state lasers and, in more detail, those for semiconductor lasers. For certain parameter values, we find a two-cluster state with either period dynamics or spiking pulsations. A discussion of the linear stability analysis of the periodic two-cluster state follows. We discuss the diffusion on the splay state branch in the presence of a noise term. We end with a short discussion.

II. MODEL

The model equations for N coupled lasers were originally proposed by Winful and Wang [10]. For the complex field amplitude X and the real normalized inversion above threshold Z of the jth laser we can write [7,10]

$$\partial_t X_j = (1 - i\alpha) Z_j X_j + \frac{i\kappa}{N} \sum_{k=1}^{N} X_k \ , \quad (1)$$

$$T \partial_t Z_j = P - Z_j - (1 + 2Z_j)|X_j|^2 \ . \quad (2)$$

In these equations, $\kappa = \kappa_R + i\kappa_I$ is the complex coupling constant [11] between the lasers, taken to be identical for every laser. P is the excess pump power above threshold and α is the linewidth enhancement factor. For solid-state lasers, $\alpha \sim 0$, while for semiconductor lasers, $\alpha \sim 5$. Time is measured in units of the photon lifetime τ_P and T is the upper level fluorescence lifetime in those units. Typical values are $T \sim 10^6$ for solid-state lasers and $T \sim 10^3$ for semiconductor lasers. If we define $\overline{X}$ as

$$\overline{X} = \frac{1}{N} \sum_{k=1}^{N} X_k, \quad (3)$$

we can rewrite (1) as

$$\partial_t X_j = (1 - i\alpha) Z_j X_j + i\kappa \overline{X} \ . \quad (4)$$

Silber *et al.* examined the linear stability of three particular states: the off state, the in-phase state, and the splay phase state. The off state corresponds to a nonlasing state of the system, i.e., $X_j = 0, Z_j = P$ ($j = 1, ..., N$). To examine the stability of the off state one introduces a small perturbation with a growth rate ω and expands around the basic state. The system loses its stability if ω becomes positive. The off state was found to lose its stability when $P = 0$ ($\kappa_I > 0$) and when $P = \kappa_I$ ($\kappa_I < 0$).

The in-phase periodic state is written as

$$X_j(t) = X_0 e^{i\Omega t} \ , \quad Z_j = Z_0 \quad (j = 1, ..., N) \ , \tag{5}$$

where

$$\Omega = \kappa_R - \alpha\kappa_I \ , \quad Z_0 = \kappa_I \ , \quad |X_0|^2 = \frac{P - \kappa_I}{1 + 2\kappa_I} \ . \tag{6}$$

In this state, the lasers are all perfectly synchronized and have a constant frequency Ω. This state exists only when $|X_0|^2 \geq 0$. The condition for stability of this state is expressed as a third-order polynomial

$$\lambda^3 + a\lambda^2 + b\lambda + c = 0 \ , \tag{7}$$

where the coefficients a, b, c are giving by [7]

$$\begin{aligned} a &= -2\kappa_I + C/T \ , \\ b &= |\kappa|^2 + \frac{2}{T}(D - C\kappa_I) \ , \\ c &= C|\kappa|^2 - 2D(\kappa_I + \alpha\kappa_R) \ , \end{aligned} \tag{8}$$

where $C = (1 + 2P)/(1 + 2\kappa_I)$ and $D = P - \kappa_I$.

The last state examined by Silber *et al.* was the splay phase state [12,13]. In this state the lasers have the same amplitude but different phases:

$$X_j(t) = X_0 e^{i\theta_j} \ , \quad Z_j = 0 \quad (j = 1, ..., N) \ , \tag{9}$$

where $|X_0|^2 = P$, i.e., this state exists only for $P \geq 0$. The crucial property of the splay state is the disappearance of the mean $\overline{X}$:

$$\sum_{k=1}^{N} e^{i\theta_k} = 0 \ . \tag{10}$$

The splay state has been observed experimentally in multimode lasers [5], in an electric oscillator circuit [14], and numerically and theoretically in other systems (Josephson junctions [15,16] and the complex Ginzburg-Landau model [8,9]).

Silber *et al.* examined only one particular splay state, which we will call the *pure* splay state. This state corresponds to the configuration where the phases are distributed periodically: $\theta_j = 2\pi j/N$. There are, however, an infinite amount of other possible configurations which satisfy the condition (10) above.

As discussed in [9], we can classify the splay states by the parameter Δ, defined as

$$\Delta = \left| \sum_{k=1}^{N} e^{2i\theta_k} \right| . \tag{11}$$

Δ can have values between 0 and 1. In the pure splay state studied by Silber *et al.*, $\Delta = 0$. The other extreme, $\Delta = 1$, corresponds to two equal size clusters of lasers with the same amplitude $|X_0|$ but with a phase shift of π.

The stability of the general splay state can be calculated using the large symmetry of the system. There are $2N - 4$ negative eigenvalues and $N - 2$ eigenvalues equal to 0. This is similar to the case in [8,9] where the number of zero eigenvalues was also $N - 2$. In both cases we have N oscillators with N arbitrary real phases subject to the condition that the mean vanishes [Eq. (10) above]. Since this complex condition gives us two constraints, we find $N - 2$ eigenvalues equal to zero.

The remaining six eigenvalues depend on the particular nature of the splay state. We find a sixth-order polynomial, which can be written as

$$P_3(\lambda)P_3^*(\lambda) - |\kappa|^2(1 + \alpha^2)\Delta^2 \frac{P^2}{T^2} = 0 \ , \tag{12}$$

where $P_3(\lambda)$ is the third-order polynomial

$$\lambda^3 + a\lambda^2 + b\lambda + c = 0 \tag{13}$$

with

$$\begin{aligned} a &= i\kappa^* + (1 + 2P)/T \ , \\ b &= [i\kappa^* + 2P(1 + i\kappa^*)]/T \ , \\ c &= i\kappa^*(1 - i\alpha)P/T \ , \end{aligned} \tag{14}$$

and $P_3^*(\lambda)$ its complex conjugate. Note that for $\Delta = 0$ this reduces to the two third-order equations found by Silber *et al.*

We can expand the stability function for large T. For the sake of brevity we will give only the result for $\Delta = 1$ and $\Delta = 0$. For $\Delta = 1$, one eigenvalue is identical to zero and for the other five we find

$$\begin{aligned} \lambda_1 &= \lambda_2^* = i\kappa + O(1/T) \ , \\ \lambda_3 &= -\left(1 + 2P + \frac{2P\kappa_I}{|\kappa|^2} + \frac{2P\kappa_R\alpha}{|\kappa|^2}\right)\frac{1}{T} + O(1/T^2) \ , \\ \lambda_4 &= \lambda_5^* = i\sqrt{\frac{2P}{T}} - \frac{1/2 + P}{T} + O(1/T^{3/2}) \ . \end{aligned} \tag{15}$$

For $\Delta = 0$, the pure splay state in [7], we find

$$\begin{aligned} \lambda_1 &= i\kappa + O(1/T) = \lambda_2^* \ , \\ \lambda_2 &= i\sqrt{\frac{2P}{T}}\sqrt{1 - i\alpha} \\ &\quad -\left(1 + 2P - \frac{iP\kappa}{|\kappa|^2} + \frac{P\kappa\alpha}{|\kappa|^2}\right)\frac{1}{T} \\ &\quad + O(1/T^{3/2}) \ , \\ \lambda_3 &= -i\sqrt{\frac{2P}{T}}\sqrt{1 - i\alpha} \\ &\quad -\left(1 + 2P + \frac{iP\kappa}{|\kappa|^2} + \frac{P\kappa\alpha}{|\kappa|^2}\right) \\ &\quad + \frac{1}{T} + O(1/T^{3/2}) \ , \end{aligned} \tag{16}$$

and where the remaining three eigenvalues are given by the complex conjugate of λ_1, λ_2, and λ_3. These expressions reduce to the ones of Silber *et al.* for small α.

From the expansions in T we see that α critically determines the region of stability of the different splay states.

In solid-state lasers, where $\alpha = 0$, the splay state loses its stability at the line $\kappa_I = 0$ for *all* values of Δ. This means that for $\kappa_I > 0$ every splay state is neutrally stable. For semiconductor lasers, where $\alpha \sim 5$, however, we see from Eqs. (15) and (16) that the splay state corresponding to $\Delta = 1$ has a larger domain in which it is neutrally stable than the splay state corresponding to $\Delta = 0$. In particular, for values of P of order 1, and thus much smaller than T, the pure splay state is unstable, while the splay state corresponding to $\Delta = 1$ is stable. The first splay state which becomes neutrally stable if one increases κ_I is a Δ not necessarily equal to one. For each value of P we have to determine this critical value of Δ.

Silber *et al.* obtained approximate regions of stability for the true splay state and the in-phase state in the limit $1/T \to 0$ and $\alpha \to 0$. In this case, corresponding to solid-state lasers, the off state is stable to the left of the lines $P = \kappa_I$ ($\kappa_I < 0$), and $P = 0$ ($\kappa_I > 0$), the in-phase state is stable to the right of the line $P = \kappa_I$ and under the line $\kappa_I = 0$, while the splay state is neutrally stable in the domain $P > 0$, $\kappa_I > 0$. In this paper, we are also interested in semiconductor laser arrays where $\alpha \sim 5$. For simplicity we have taken $\alpha = 5$, $T = 10^3$, and $\kappa_R = 1$ for semiconductor lasers and $\alpha = 0$, $T = 10^6$, and $\kappa_R = 1$ for solid state lasers.

In Fig. 1 we have plotted the lines of stability for the in-phase state and the splay phase state for semiconductor lasers. There are three regions where one of the simple states described above are neutrally stable. The dash-dotted line limits the stability region of the off state. The solid line denotes the lower boundary of the neutrally stable domain of the splay state. Note that the splay state exists only for $P \geq 0$. The dashed line defines the stability boundary for the in-phase state. In the region between the dashed line and the solid line there is no simple stable solution. Note that this is not the case for the solid-state lasers [7], where there is always a neutrally stable simple state. For the range of parameter values in Fig. 1, only splay states with $\Delta \approx 1$ are neutrally stable. For smaller values of Δ, the splay state is unstable.

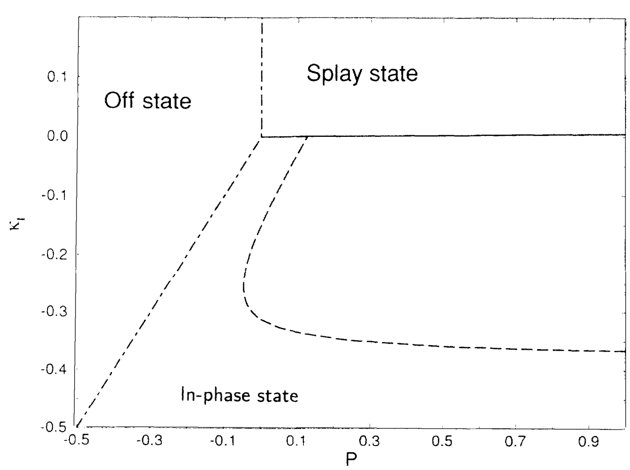

FIG. 1. The phase diagram for semiconductor lasers. The regions of stability for the three states are indicated. Note that there is a domain in which none of the three states are stable.

III. NUMERICAL RESULTS

The model was integrated numerically using as initial conditions a random distribution in the square $|X| < 1$ and on the line $0 < Z < 1$. We have verified that changing the initial conditions does not affect the results. We have simulated the equations for the two systems $\alpha = 0$, $T = 10^6$ and $\alpha = 5$, $T = 10^3$.

For solid-state lasers all initial conditions studied here and for all values of N, we find a final state corresponding to the phase diagram. If one starts from random initial conditions in the region where the off state or the in-phase state is stable, the final state will be the off state or the in-phase state. If we start in the region where the splay state is stable, the final result will be a splay state with a value of Δ between 0 and 1. In Fig. 2 we have plotted in the complex plane the values of X_j after such a run ($N = 10$, $\kappa_I = 0.2$ and $P = 0.01$). The value for this particular run is $\Delta = 0.20$.

For semiconductor lasers, as for solid-state lasers, we find that if we start from random initial conditions and parameter values for which the off state or in-phase state is stable, the final state is the off state and the in-phase state, respectively.

In the region where the splay state is neutrally stable, however, the system does not always relax to this state. In fact, only for physically unrealistic values of P, i.e., P of the order of T, do we find that random initial conditions give us the splay state [18].

For smaller values of P, the system is chaotic. Each laser describes a chaotic trajectory and the mean $\overline{X}$ is of the order $1/\sqrt{N}$. In Fig. 3 we have plotted the time series of one laser and Fig. 4 shows its power spectrum for $\kappa_I = 0.2$, and $P = 0.01$. For these values only the splay states with $0.979 \leq \Delta \leq 1$ are neutrally stable. The splay states with other values of Δ are unstable. Apparently, the basin of attraction for the stable splay states belonging

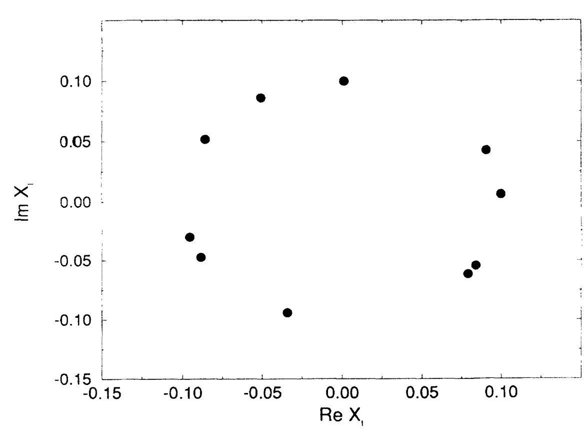

FIG. 2. The values of X_j for $T = 10^6$, $\alpha = 0$, $P = 0.01$, $\kappa_I = 0.2$, and $N = 10$. The final state is a splay state (i.e., $\overline{X} = 0$) and Δ was found to be approximately 0.2.

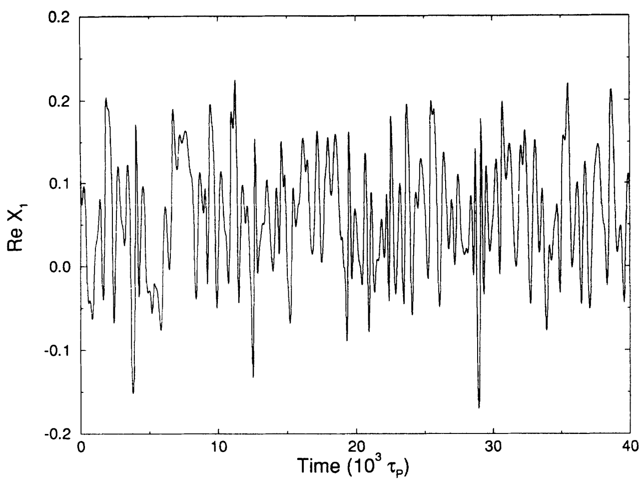

FIG. 3. Time series of the real part of X_1 ($N = 10$, $P = 0.01$, and $\kappa_I = 0.2$).

to this narrow range of Δ is too small, never allowing the system to relax to the splay state, unless we start with initial conditions very close to a stable splay state.

The narrow range of allowable splay states can have important implications. If we have a neutrally stable splay state for all values of Δ, it has been shown numerically in a model describing globally coupled Josephson junctions [15] that adding a noise term to the equations will result in a diffusion on the solution branch of the splay state. This was termed interhyperhedral diffusion in [15]. To investigate this diffusion in our model we have added a random term $\epsilon\eta_j(x,t)$ to Eq. (1), where ϵ is the noise strength and $\eta_j(x,t)$ are δ-correlated random functions with zero width and unit variance. For $\alpha = 0$, the splay state is neutrally stable for all values of Δ and adding a noise term leads to a diffusion on the whole solution branch. This can be seen in Fig. 5.

For $\alpha = 5$ however, such a diffusion cannot occur. The splay states are only neutrally stable over a very small range of Δ, which will prevent the system from wandering over the whole branch. Indeed, numerically we have found that adding a noise term to the equations will lead to a diffusion confined to the very small neutrally stable part of the branch, as can be seen in Fig. 6. If the noise is too strong the system will be kicked out of the splay state and will become chaotic.

In the region where neither one of the three states described in Sec. II is stable, the dynamics of the system depends critically on the number of lasers N. We will not give an exhaustive and quantitative analysis of the dynamics but will limit ourselves to a general description of the observed behavior. For small values of N, typically less than 10, we find that there is a region in which the system relaxes to two clusters with population N_1 and $N_2 = N - N_1$, respectively. The two clusters have different amplitudes but the same frequency. This state loses

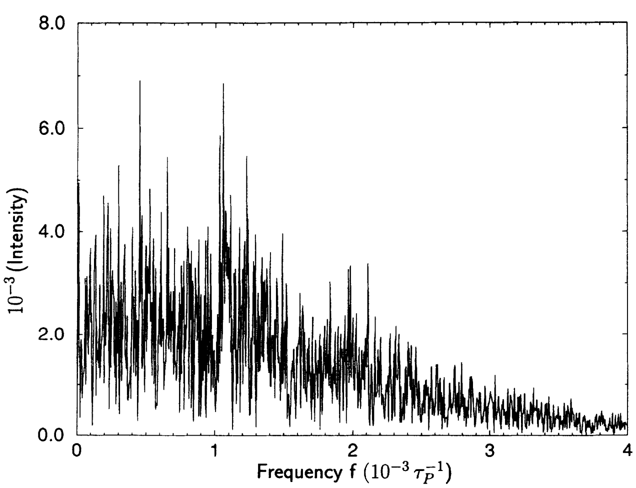

FIG. 4. The power spectrum of the time series in Fig. 3.

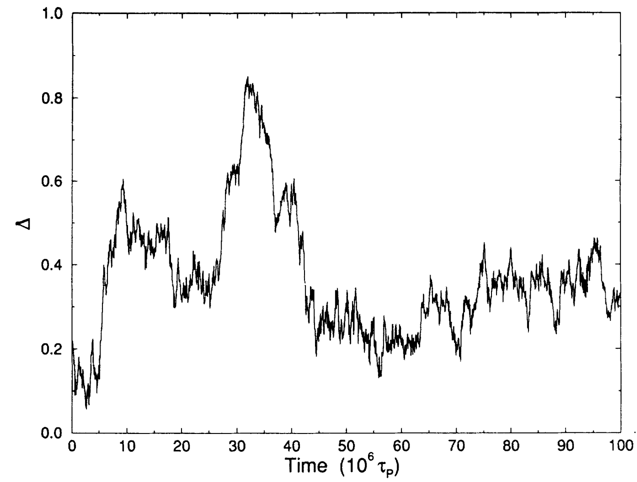

FIG. 5. Δ as a function of time for the splay state with the presence of a noise term. The parameters are the same as in Fig. 2, with $\epsilon = 1 \times 10^{-4}$.

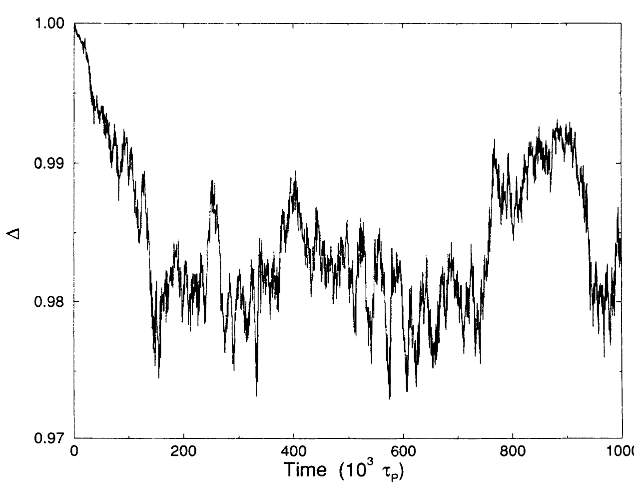

FIG. 6. Δ as a function of time for the splay state with the presence of a noise term. The parameters are the same as in Fig. 2, with $\epsilon = 1 \times 10^{-4}$.

its stability to a state where the clusters exhibit spiking pulsations (see below).

We can determine the amplitudes, phase shift, and frequency of the two clusters numerically. Since the clusters have periodic dynamics we can describe the two clusters by

$$\begin{aligned} X_1 &= R_1 e^{i\Omega t+\phi_1} , \\ X_2 &= R_2 e^{i\Omega t+\phi_2} , \\ \overline{X} &= [(1-\rho)R_1 e^{i\phi_2} + \rho R_2 e^{i\phi_2}]e^{i\Omega t} , \\ Z_i &= (P - R_i^2)/(1+2R_i^2) \quad (i=1,2) , \end{aligned} \tag{17}$$

where $\rho = 1/N_2$. Substituting this into the equations for X_1 and X_2 we get four coupled nonlinear equations for R_1, R_2, Ω, and the phase difference $\Delta\phi \equiv \phi_2 - \phi_1$. These equations can be solved numerically to find the cluster state.

Once we have the values of R_1, R_2, Ω and $\Delta\phi$ we can calculate the stability of the two-cluster state. We replace R_j and Z_j (j=1,...,N) in the full set of equations [Eqs. (1) and (2)] by $R_j+\delta_j$ and Z_j+z_j where δ_j is complex, z_j is real and $|\delta_j|, |z_j| << 1$. After linearizing the equations the resulting $3N \times 3N$ stability matrix can be solved. We find two third-order polynomials with multiplicity $N_1 - 1$ and $N_2 - 1$, respectively, which can be written as

$$\lambda^3 + a\lambda^2 + b\lambda + c = 0 , \tag{18}$$

where the coefficients a, b, c are given by

$$\begin{aligned} a &= -2Z_i + r_i , \\ b &= (\Omega + \alpha Z_i)^2 + Z_i^2 + r_i R_i - 2s_i Z_i , \\ c &= s_i(\Omega + \alpha Z_i)^2 - r_i R_i(\alpha\Omega + Z_i + \alpha^2 Z_i) + s_i Z_i^2 , \end{aligned} \tag{19}$$

where $r_i = 2R_i(1+2Z_i)/T$, $s_i = (1+2R_i^2)/T$, and $i = 1,2$ correspond to clusters 1 and 2, respectively. Note that the multiplicity comes from the perturbations inside the cluster with N_1 and N_2 lasers, respectively. Of the remaining six eigenvalues, one is identical to 0, due to the invariance of the system to a uniform phase shift, and the other five are given by the roots of a fifth-order expression which is too lengthy to display here.

As an example of the dynamics of the cluster state we have taken as parameter values $\kappa_I = -0.2$ and $P = 0.01$. Analytically, we find that all cluster states are unstable if $N \geq 10$. Furthermore, for smaller values of N, the only periodic cluster state which is stable is the state in which $N-1$ lasers are in one cluster and the remaining one in the other. Indeed, numerically we find a stable periodic two-cluster state ($N_1 = 8$) if we start with nine lasers but a nonperiodic two cluster state if we start with ten lasers ($N_1 = 9$). As an example of a case in which the periodic two cluster state is stable, we have drawn the region corresponding to $N = 6$ in Fig. 7. The periodic two cluster state is stable in the region marked "Cluster." All the lines are determined analytically.

The dynamics in the case where the periodic two cluster state is unstable is shown in Fig. 8, where we have discarded the transients. The system remains in a two-cluster state, but the clusters now exhibit a spiking pulsation. This type of spiking oscillation has also been found

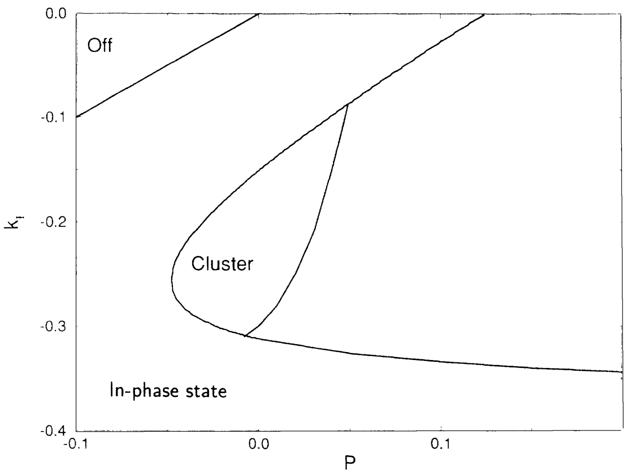

FIG. 7. The phase diagram for $N = 6$, showing the region in which the two cluster state is stable.

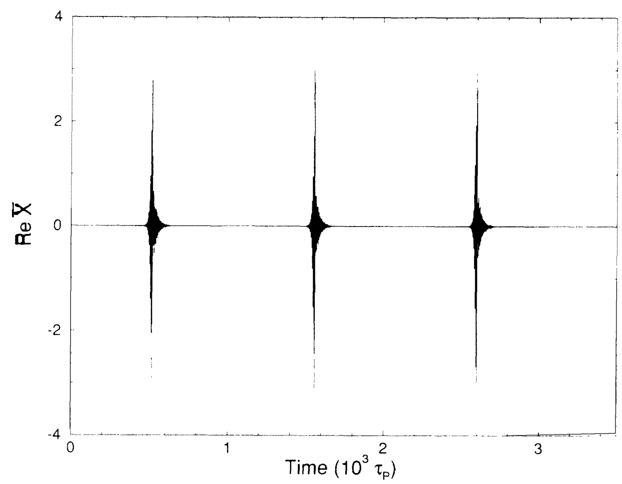

FIG. 8. The time series of $\overline{X}$ for $\kappa_I = -0.2$, $P = 0.01$ and $N = 10$.

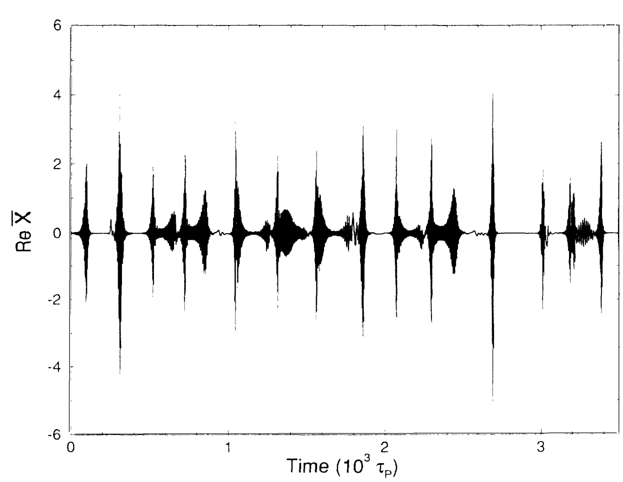

FIG. 9. The time series of X_1 for $\kappa_I = -0.2$, $P = 1$ and $N = 10$.

in a numerical investigation of a model of modulated multimode lasers [17]. The pulses are separated by periods of the order of T. During this period the complex amplitude is very small; $X << 1$ and Z grows exponentially to the value corresponding to the off state. The off state, however, is not stable and $|X|$ starts to grow. We then have a period in which X oscillates with a frequency of the order $1/T$, Z decreases rapidly and $|X|$ pulses.

For larger values of P, the system no longer settles into a two-cluster state and the dynamics is more complicated. A time series for the case $\kappa_I = -0.2$ and $P = 1$ is shown in Fig. 9. Remnants of the pulsations of Fig. 8 remain, remain, but the global dynamics is chaotic.

IV. DISCUSSION

In this paper we have investigated in more detail a model of globally coupled lasers recently discussed by Silber *et al.* We have determined the linear stability for the *general* splay state, which can be characterized by a parameter Δ with values between 0 and 1. We have shown that, depending on the type of laser, there is either a very narrow window of neutrally stable Δ or a very large one. This can have important implications, since the presence of a noise term leads to diffusion on the solution branch.

For semiconductors lasers, there is a region in the parameter space where none of the simple states are stable. Numerically we have found that the system relaxes in a certain region of parameter space to a periodic two-cluster state. We have calculated analytically the stability of this cluster state and found that it depends critically on the number of lasers in the system.

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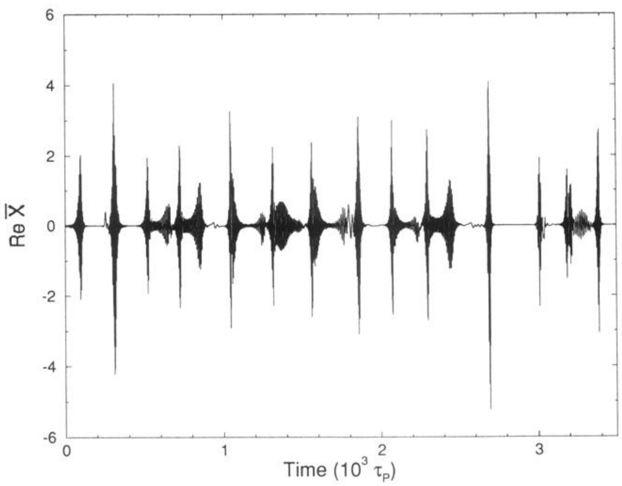

FIG. 9. The time series of X_1 for $\kappa_I = -0.2$, $P = 1$ and $N = 10$.